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ON THE RESIDUE OF THE SPECTRAL ZETA FUNCTIONS OF KÄHLER MANIFOLDS WITH CONICAL SINGULARITIES

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0. Introduction

Let $\pi : M \to B$ be a family of Kähler manifolds, and $p : \xi \to M$ a holomorphic vector bundle with a Hermitian metric. Then, from the work of Quillen, the Knudsen-Mumford determinant $\lambda(\xi)$ admits a canonical Hermitian metric called the Quillen metric. In [9], [10], [11], Bismut, Gillet and Soulé calculated the curvature of $\lambda(\xi)$ and obtained the refinement of Grothendieck-Riemann-Roch theorem. In [3], their result was generalized to the case of degenerating family of Riemman surfaces by Bismut and Bost. But there is no result on the curvature of Knudsen-Mumford determinant for family of Kähler manifolds with boundary or singular Kähler manifolds.

As for the real case, in [4] - [6], Bismut and Cheeger extended the result of Atiyah-Patodi-Singer on the index of the Dirac operator on manifolds with boundary. They patched a cone to the boundary of the manifold and considered a manifold with conical singularities. They gave a detailed study of elliptic operators on such singular manifolds and obtained the family index theorem.

To consider the extension of the formula in [4], [5] in the direction of [7], [8] and [9] - [11], it is necessary to define the Quillen metric for the family of manifolds with conical singularities. Therefore we must consider the Ray-Singer analytic torsion on manifolds with conical singularities. By definition, it is given by a certain sum of the derivative at the origin of spectral zeta functions. From the results of Cheeger, these zeta functions possibly have a simple pole at the origin. Thus it is not clear whether the analytic torsion is defined for them.

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The purpose of this article is to establish a relation among the residue of zeta functions at the origin and to show that the analytic torsion is defined for certain singular Kähler manifolds.

Let (M, g) be a compact Kähler manifold with an isolated singularity p. We say that p is a conical singularity if there is an open neighborhood U of p, a metric cone $X = C(N)$ on a compact Riemannian manifold (N, g_N) and a map $i : U \to C^*_{0,1}(N)$ such that i is an isometry between the smooth part of U and $C^*_{0,1}(N)$; i.e.,

$$i^* g_X = g|_U, \quad g_X = dr^2 + r^2 g_N. \tag{0.1}$$

In the above definition, $C^*_{0,r}(N)$ is the metric completion of $C_{0,r}(N) = (0, r) \times N$. We say that $X = C(N)$ is the model cone of the singularity p.

We consider the following special and important case. Let $\pi : L \to Y$ be a negative line bundle over a compact projective algebraic manifold Y. When L is negative, we write $L < 0$. Then we can contract the zero section of L, denoted by Z_Y, and obtain a new space X which may possibly have an isolated singularity p. We say that X is the Stein reduction of L. Since L admits a $\mathbb{C}^*$-action defined by

$$T_\lambda(\zeta) := \lambda\zeta, \tag{0.2}$$

X admits the induced $\mathbb{C}^*$-action which is also denoted by T_λ. Let g_X be a Kähler metric on X. We say that g_X is a conical metric if it satisfies the following condition: there is a positive integer $a \in \mathbb{Z}_+$ such that

$$T^*_\lambda g_X = |\lambda|^{2a} g_X \tag{0.3}$$

for every $\lambda \in \mathbb{C}^*$. If g_X is a conical metric, then by setting $N := \{x \in X; dist_X(p, x) = 1\}$, where $dist_X(\cdot, \cdot)$ is the distance function on X, we have the following expression:

$$g_X = dr^2 + r^2 g_N, \quad g_N := g|_N. \tag{0.4}$$

Definition 0.1. Let (M, g) be a compact Kähler manifold with an isolated singularity p. We say that p is a conical singularity associated to a line bundle $\pi : L \to Y$ if $L < 0$ and if the model cone X of the singularity p is the Stein reduction of L whose metric is a conical Kähler metric g_X.

For a Riemannian manifold (M, g), we denote by σ_M the scaler curvature and by Ric_M the Ricci curvature. Define $\lambda(x)$ and $\lambda_+(x)$ by

$$(0.5)\qquad \lambda(x) := \sup_{\xi \in T_x M - \{0\}} \frac{\mathrm{Ric}_{M,x}(\xi, \xi)}{g_x(\xi, \xi)}, \quad \lambda_+(x) := \max\{\lambda(x), 0\}.$$

We can now state our main theorem.

Main Theorem. *Let (M, g) be an n-dimensional compact Kähler manifold with a conical singularity associated to a line bundle $\pi : L \to Y$. Let (X, g_X) be the Stein reduction of L with a conical Kähler metric. If $K_Y - aL < 0$ where a is the same integer as (0.3) and*

$$(0.6)\qquad \inf_{X - \{p\}} r^2(\sigma_X - \lambda_+) > -(n-1)^2,$$

then the following equality holds:

$$(0.7)\qquad \sum_{q=0}^{n} (-1)^q q \cdot \mathrm{Res}_{s=0}\, \zeta_{0,q}(s) = 0,$$

where $\zeta_{0,q}(s)$ is the spectral zeta function of $\Box_{0,q}$, the Friedrichs extension of the Laplacian on $(0, q)$-forms on M. When $n = 1$ and 2, (0.7) holds for every Kähler manifold with conical singularities (cf. [29]).

From a theorem of Cheeger, each $\zeta_{0,q}(s)$ has at most a simple pole at the origin. Therefore from the Main Theorem, we have the following corollary.

Corollary 0.1. *Let (M, g) be the same as in the Main Theorem. Then we can define its analytic torsion by the following formula:*

$$(0.8)\qquad T(M, g) := \exp\Big(-\frac{d}{ds}\Big|_{s=0}\Big(\sum_{q=0}^{n} (-1)^q q \zeta_{0,q}(s)\Big)\Big).$$

Our Main Theorem is a special case of the following theorem.

Theorem 0.1. *Let (M, g) be a compact Kähler manifold with a conical singularity p whose model cone is (X, g_X). Let $\tilde{X}$ be a desingularization of X, and set*

$$(0.9)\qquad \mathcal{H}_q(\tilde{X}) := \left\{ f \in \Omega_q(\tilde{X});\ \int_{\tilde{X}} \frac{|f|^2}{1 + r^2} dv < \infty,\ \int_{\tilde{X}} |f|^{\frac{2n}{n-1}} dv < \infty \right\}$$

where $\Omega_q(\tilde{X})$ is the space of holomorphic q-forms on $\tilde{X}$. Assume that Y, the exceptional divisor of $\tilde{X}$, is smooth. If (0.6), $\mathcal{H}_q(\tilde{X}) = 0$

$(0 \leq q \leq n)$ *and* $H^0(Y, \Omega_Y^q) = 0$ $(0 < q < n)$ *hold, then* (0.7) *holds for* (M, g).

As the referee pointed out, it is expected that the formula (0.7) holds for every Kähler manifold with conical singularities without various assumptions in the Main Theorem, by using the local index cancellation formula as in section 7. We also remark that the formula (0.7) holds for the vector bundle case under the semi-positivity condition of the bundle.

This article is arranged as follows. In section 1, we define a conic degenerating family of Kähler manifolds for a given Kähler manifold with a conical singularity. In section 2, we establish an estimate of the heat kernels needed below. In section 3, we prove the Hardy and Sobolev inequality on cones. In section 4, we prove the uniformity of the asymptotic expansion of the trace of the heat kernels for the family in section 1. In section 5, we prove the uniform lower bound of the first eigenvalue of the Laplacians for the family. In section 6, we prove the Main Theorem. Our proof is given as an application of our previous results (cf.[27, Theorem B]), Cheeger's Theorem (cf. [13]) and the theorem of [9], [10], [11]. In section 7, we treat the 2-dimensional cases. In section 8, we shall show that Kähler manifolds with nodes are examples for which the Main Theorem holds.

1. A conic degenerating family of Kähler manifolds

Let (M, g) be a compact Kähler manifold with a conical singularity p of dimension n; i.e., there is a neighborhood U of p and an identification such that

$$(U, g) = (C_{0,1}^*(N), dr^2 + r^2 ds_N^2) \tag{1.1}$$

for some compact Riemannian manifold (N, ds_N^2). We assume that $X := (C(N), dr^2 + r^2 ds_N^2)$ is a Kähler manifold whose homothetic transformation is holomorphic; i.e., $T_\lambda(r, x) := (\lambda r, x)$ is a holomorphic isomorphism on X where (r, x) is the polar coordinate of $X = C(N)$. Note that $T_\lambda^* g_X = \lambda^2 g_X$ where $g_X = dr^2 + r^2 ds_N^2$. Let $\pi : \tilde{X} \to X$ be a desingularization, and $g_{\tilde{X}}$ be a Kähler metric on $\tilde{X}$ such that on $\tilde{X} - \pi^{-1}(C_{0,1}^*(N))$,

$$g_{\tilde{X}} = \pi^* g_X, \quad g_X = dr^2 + r^2 ds_N^2. \tag{1.2}$$

Set $M' := M - C^*_{0,1}(N)$. Then $\tilde{M} := M' \cup_N \pi^{-1}(C^*_{0,1}(N))$ is a desingularization of M. Define a family of Kähler metrics $\{g_\epsilon\}$ by

$$g_\epsilon(x) = \begin{cases} \epsilon^2 g_{\tilde{X}}(x), & x \in \pi^{-1}(C^*_{0,\epsilon^{-1}}(N)), \\ g(x), & x \in M', \end{cases} \tag{1.3}$$

where we use the identification $T_\epsilon : C^*_{0,\epsilon^{-1}}(N) \to C^*_{0,1}(N)$ to patch $\pi^{-1}(C^*_{0,\epsilon^{-1}}(N))$ and M'. Since T_ϵ induces an isometry between $(C_{\frac{1}{2}\epsilon^{-1},\epsilon^{-1}}(N), \epsilon^2 g_X)$ and $(C_{\frac{1}{2},1}(N), g_X)$, g_ϵ is a smooth Kähler metric on $\tilde{M}$, and $(\tilde{M}, g_\epsilon)$ converges to (M, g) as $\epsilon \to 0$. We remark that $\{(\tilde{M}, g_\epsilon)\}$ is a conic degeneration in the sense of [27].

Set $p : \mathcal{M} := \tilde{M} \times \Delta^* \to \Delta^*$ where $\Delta = \{z \in \mathbb{C}; |z| < 1\}$ and $\Delta^* = \Delta - \{0\}$. Then $T\tilde{M}$ is a holomorphic subbundle of $T\mathcal{M}$. Let G be a Hermitian metric on $T\tilde{M}$ defined by

$$G|_{T\tilde{M}_\epsilon} = g_{|\epsilon|}, \tag{1.4}$$

where $\tilde{M}_\epsilon := p^{-1}(\epsilon)$ for $\epsilon \in \Delta$. Denote by $R(T\tilde{M}, G)$ the curvature of $(T\tilde{M}, G)$ with respect to the Hermitian connection. Then $R(T\tilde{M}, G) \in A^{1,1}(\mathcal{M}, \operatorname{End} T\tilde{M})$. Let $\operatorname{Td}(R(T\tilde{M}, G)) \in \oplus_p A^{p,p}(\mathcal{M})$ be the Todd form. Then $\int_{\mathcal{M}/\Delta} \operatorname{Td}(R(T\tilde{M}, G)) \in \oplus_{p\le 2} A^{p,p}(\Delta^*)$, where $\int_{\mathcal{M}/\Delta}$ implies the integration along the fiber. The following proposition is needed for the proof of the Main Theorem.

Proposition 1.1. $\left[\int_{\mathcal{M}/\Delta} \operatorname{Td}(R(T\tilde{M}, G))\right]^{(1,1)}$ *can be extended to a smooth (1,1)-form on* Δ *where* $[\omega]^{(p,p)}$ *denotes the degree (p,p)-part of* ω.

Proof. Since $\dim \mathcal{M} = n + 1$, we have

$$\left[\int_{\mathcal{M}/\Delta} \operatorname{Td}(R(T\tilde{M}, G))\right]^{(1,1)} = \int_{M_\epsilon} \left[\operatorname{Td}(R(T\tilde{M}, G))\right]^{(n+1,n+1)}. \tag{1.5}$$

Set $\left[\operatorname{Td}(R(T\tilde{M}, G))\right]^{(n+1,n+1)} = A \wedge d\epsilon \wedge d\bar{\epsilon}$ where A is a relative (n,n)-form on $\mathcal{M}$. Since $g_\epsilon = g$ on M', it is clear that $\int_{M'} A|_{M_\epsilon}$ is extended to a smooth (1,1)-form on Δ. Therefore it is sufficient to show that $\int_{\pi^{-1}(C^*_{0,1}(N))} A|_{M_\epsilon}$ extends to a smooth form on Δ.

Set $\mathcal{X} := \tilde{X} \times \Delta^*$ and $T\tilde{X} := Ker(p_2)$ where $p_2 : \mathcal{X} \to \Delta^*$ is the projection. Consider two Hermitian metrics on $T\tilde{X}$, $g_{\tilde{X}}$ and $\tilde{G} := |\epsilon|^2 g_{\tilde{X}}$.

Denote by $R(T\tilde{X}, \tilde{G})$ (resp. $R(T\tilde{X}, g_{\tilde{X}})$) the curvature of $(T\tilde{X}, \tilde{G})$ (resp. $(T\tilde{X}, g_{\tilde{X}})$) with respect to the Hermitian connection. By computation

$$(1.6) \qquad R(T\tilde{X}, \tilde{G}) = R(T\tilde{X}, g_{\tilde{X}}).$$

Therefore,

$$(1.7) \qquad \left[\mathrm{Td}(R(T\tilde{X}, \tilde{G}))\right]^{(n+1,n+1)} = 0,$$

since every polynomial of $R(T\tilde{X}, g_{\tilde{X}})$ has no component of degree (n+1,n+1).

Set $\mathcal{Y} := \pi^{-1}(C^*_{0,1}(N)) \times \Delta^* \subset \mathcal{M}$ and consider $T\tilde{X}|_{\mathcal{Y}}$. Define an embedding $T : \mathcal{Y} \to \mathcal{X}$ by $T(x, \epsilon) := (T_{|\epsilon|^{-1}}(x), \epsilon)$. Thus $T^*\tilde{G} = G|_{\mathcal{Y}}$, which gives

$$(1.8) \quad \left[\mathrm{Td}(R(T\tilde{M}, G))\right]^{(n+1,n+1)}\Big|_{\mathcal{Y}} = T^*\left(\left[\mathrm{Td}(R(T\tilde{X}, \tilde{G}))\right]^{(n+1,n+1)}\right).$$

From (1.7), the right-hand side of (1.8) vanishes. Therefore $\int_{\pi^{-1}(C^*_{0,1}(N))} A|_{M_\epsilon} = 0$, which completes the proof.

2. Heat kernels for Schrödinger operators on asymptotically flat vector bundles

In this section, we shall generalize the result obtained in [27, §2] to the cases of Schrödinger operators on certain vector bundles.

Let (X, g) be a complete Riemannian manifold of dimension $m = 2n$. We fix a point o in X, and set $|x| := dist(o, x) = d(o, x)$. Let i_x be the injectivity radius at x.

Definition 2.1. Let (X, g) and o be as stated above. We say that (X, g) is an (pointed) asymptotically flat manifold if the following two conditions are satisfied.

There is a constant $c > 0$ such that for all $y \in X$, $i_y \geq c(1 + |y|)$.

Set $j_y := c(1 + |y|)$. Let $B(y, j_y)$ be the metric ball of radius j_y centered at y, and $x = (x^1, \cdots, x^m)$ the geodesic normal coordinate on $B(y, j_y)$. If we write $g(x) = \sum_{i,j} g_{ij}(x) dx^i dx^j$ on $B(y, j_y)$, then

$$C_0^{-1} I \leq (g_{ij}(x)) \leq C_0 I, \quad \left|\frac{\partial^\alpha}{\partial x^\alpha} g_{ij}(x)\right| \leq K_\alpha (1 + |y|)^{-|\alpha|}$$

hold for all $x \in B(y, j_y)$ and $\alpha > 0$, where C_0 and K_α are constants independent of x, y. When $y = o$, we simply write $B(r)$ instead of $B(o, r)$.

Assumption 2.1. *Let (X, g) be an asymptotically flat manifold. We assume the followings.*

1) *There is a constant $D > 0$ such that for all $f \in C_0^\infty(X)$, $\|df\|_{L^2} \geq D \cdot \|f\|_{L^{\frac{2m}{m-2}}}$ if $m > 2$ and the same inequality for $(X \times \mathbb{C}, g + ds_e^2)$ if $m = 2$.*
2) *There is a constant $A > 0$ such that $Area(S(r)) \leq A \cdot r^{m-1}$ for all $r \geq 0$, where $S(r) := \{x \in X; |x| = r\}$.*

Throughout this article, we assume that Assumption 2.1 is always satisfied for asymptotically flat manifolds

Let (E, h, ∇^E) be a Hermitian vector bundle of rank r with a Hermitian connection on X. Since E is trivial on each $B(y, j_y)$, we can choose a unitary frame $\{s_1, \cdots, s_r\}$. With respect to this frame, we set

$$h_{ij}(x) := h(s_i, s_j)(x), \quad \nabla^E s_i(x) = \sum_j \omega_{ij}(x) s_j(x).$$

We denote by $\Omega = (\Omega_{ij}) = d\omega + \omega \wedge \omega$ the curvature form of E.

Definition 2.2. We say that (E, h, ∇^E) is an asymptotically flat vector bundle on X if for every $y \in V$, there is a suitable choice of frame $\{s_1, \cdots, s_r\}$ on $B(y, j_y)$ such that $h_{ij}(y) = \delta_{ij}$ and

$$C^{-1} I \leq (h_{ij}(x)) \leq CI, \quad |\frac{\partial^\alpha}{\partial x^\alpha} h_{ij}(x)| \leq K_\alpha (1 + |y|)^{-|\alpha|},$$

$$|\frac{\partial^\beta}{\partial x^\beta} \omega_{ij}(x)| \leq K_\beta (1 + |y|)^{-(|\beta|+1)}$$

for all $x \in B(y, j_y)$, $y \in X$ and $\alpha, \beta \geq 0$ where C and K_α are constants independent of x, y.

Let $\Delta^E := \nabla^{E*} \cdot \nabla^E$ be the Bochner Laplacian on E. Then, for $F \in C^\infty(X, \mathrm{Herm}(E))$, $H := \Delta^E + F$ is a self-adjoint Schrödinger opetator on E.

Definition 2.3. Let (E, h, ∇^E) be an asymptotically flat vector bundle on X. We say that F is an asymptotically flat potential if

$$|\frac{\partial^\alpha}{\partial x^\alpha} F_{ij}(x)| \leq K_\alpha (1 + |y|)^{-(|\alpha|+2)}$$

holds for all $x \in B(y, j_y)$, $y \in X$ and $\alpha \geq 0$ where K_α are constants independent of x, y.

Now we shall consider the heat kernel for the Schrödinger operator $H := \Delta^E + F$ where F is an asymptotically flat potential. We denote it by $K^E(t, x, y; H)$. When $F = 0$, we simply write $K^E(t, x, y)$. Then, the parabolic Harnack inequality of Li and Yau, combined with a theorem of Hess, Schrader and Uhlenbrock, gives an upper bound of the heat kernel on each $B(y, j_y)$.

Lemma 2.1. *Let $q = q(x) \in C^\infty(X)$ be an asymptotically flat potential and $H = \Delta + q$ be a self-adjoint Schrödinger operator on $L^2(X)$. Then, for every $p \in X$, $x, y \in B(p, j_p)$ and $0 \leq t \leq \frac{1}{4}j_p^2 = \frac{1}{4}c^2(1 + |p|)^2$, the following estimate holds for $K(t, x, y; H)$:*

$$K(t, x, y; H) \leq Ct^{-n} \exp(-\frac{\gamma d(x, y)^2}{t}),$$

where C and γ are positive constants independent of p, x, y, t.

Proof. We can prove the above estimate using Theorem 3.3 and Corollary 3.1 of [22], noting that asymptotical flatness implies $A \leq CR^{-2}$ in Theorem 3.3 of [22].

Now we consider the vector bundle case. Let (E, h, ∇^E) and F be the same as in Definitions 2.2, 2.3 and $H = \Delta^E + F$ be a self-adjoint Schrödinger operator. Then, we have the following proposition.

Proposition 2.1. *For every $p \in X$, $x, y \in B(p, j_p)$ and $0 \leq t \leq \frac{1}{4}j_p^2$,*

$$|K^E(t, x, y; H)| \leq Ct^{-n} \exp(-\frac{\gamma d(x, y)^2}{t}), \tag{2.1}$$

where C and γ are positive constants independent of p, x, y, t. Here the norm $|\cdot|$ is the operator norm on $\mathrm{Hom}(E_y, E_x)$.

Proof. Since F is asymptotically flat, there is an asymptotically flat potential $q \in C^\infty(X)$ such that $q(x) \geq 0$ and $-q(x)I_E \leq F(x) \leq q(x)I_E$ for $x \in X$. Then, by a theorem of [19, §3], setting $H' := \Delta - q$, we have the following estimate:

$$|K^E(t, x, y; H)| \leq K(t, x, y; H')$$

for all $(t, x, y) \in (0, \infty) \times X \times X$. Therefore (2.1) is an immediate consequence of Lemma 2.1 and the above inequality.

To study the asymptotic expansion, we need a good parametrix. Following [2], we identify E with $E\otimes|\Lambda|^{\frac{1}{2}}$ where $|\Lambda^{\frac{1}{2}}|$ is the half density bundle on X. We construct a parametrix as follows, (cf. [2, pp.82-87]).

Let $x = (x^1,\cdots,x^m)$ be the geodesic normal coordinate centered at y. In these coordinates, the metric tensor is represented by $g = \sum_{ij} g_{ij}(x)dx^i dx^j$. Set $\theta(x,y) = \det(g_{ij}(x))^{\frac{1}{2}}$, and define a differential operator B by

$$B := \theta^{\frac{1}{2}} \circ H \circ \theta^{-\frac{1}{2}}. \tag{2.2}$$

Let $\tau(x,y) : E_y \to E_x$ be the parallel transport along the geodesic joining y and x. Then, on $B(y,j_y)$, we can write $\tau(x,y) = (\tau_{ij}(x,y))$ with respect to the frame in Definition 2.2. It is easily verified that $C^{-1}I \le (\tau_{ij}(x,y)) \le CI$ and each $\tau_{ij}(x,y)$ satisfies the following decay condition for $\alpha \ge 0$.

Lemma 2.2.

$$|\frac{\partial^\alpha}{\partial x^\alpha}\tau_{ij}(x,y)| \le K_\alpha(1+|y|)^{-|\alpha|}. \tag{2.3}$$

Proof. Since E is trivial on $B(y,j_y)$, we may represent $\tau(x,y)$ by a matrix $P(x)$. Set $P(t,x) := P(\exp_y(t\xi))$, where $x = \exp_y(\xi d(x,y))$. Thus $P(x) = P(d,x)$, $d = d(x,y)$. Thus $P(t,x)$ satisfies the following ordinary differential equation.

$$\frac{d}{dt}P(t,x) + \omega(t\xi)P(t,x) = 0, \quad P(0,x) = I.$$

Using the above equality, we have

$$\frac{d}{dt}|P(t)|^2 = -<\omega(t)P(t),P(t)> - <P(t),P(t)\omega(t)>.$$

Therefore, $\frac{d}{dt}|P(t)| \le |\omega(t)|\cdot|P(t)|$, which implies that

$$|P(t)| \le \exp(\int_0^t |\omega(s)|ds).$$

Since ω satisfies the decay condition (cf.Definition 2.2), we have $|P(x)| \le K$, where K is a constant independent of x, y. In the same way, we can prove $|P(x)^{-1}| \le K$. This proves the lemma when $\alpha = 0$.

When $\alpha > 0$, we prove by induction. We assume $|\nabla_x^{k-1} P(t,x)| \leq C_{k-1}(1+|y|)^{-(k-1)}$. Then

$$|\frac{d}{dt}\nabla^k P(t) + \omega(t)\nabla^k P(t)| \leq C_k(1+|y|)^{-(k+1)}.$$

From this, we have

$$\frac{d}{dt}|\nabla^k P(t)|^2 \leq 2|\omega(t)| \cdot |\nabla^k P(t)|^2 + \frac{C_k}{(1+|y|)^{k+1}}|\nabla^k P(t)|.$$

This implies the following inequality

$$\frac{d}{dt}|\nabla^k P(t)| \leq \frac{C_0}{(1+|y|)}|\nabla^k P(t)| + \frac{C_k}{(1+|y|)^{k+1}},$$

which in turn gives

$$\frac{d}{dt}\log(\frac{C_0}{(1+|y|)}|\nabla^k P(t)| + \frac{C_k}{(1+|y|)^{k+1}}) \leq \frac{C_0}{(1+|y|)}.$$

Since we can easily show that $|\nabla_x^k P(t,x)|_{t=0}| \leq C_k(1+|y|)^{-k}$, we have

$$|\nabla^k P(t)| \leq \frac{C_k}{(1+|y|)^k}\{1 + \exp(\frac{C_0 t}{1+|y|})\}.$$

Setting $t = d(x,y)$, we obtain the desired inequality for $\nabla^k P(x)$.

Now, we define functions $u_i(x,y;H)$ on $B(y,j_y)$ inductively by

$$u_0(x,y;H) := \tau(x,y) \tag{2.4}$$

$$\frac{u_i(x,y;H)}{\tau(x,y)} := -\int_0^1 \frac{s^{i-1}B_x u_{i-1}(x_s,y;H)}{\tau(x_s,y)}ds, \tag{2.5}$$

where $x_s = \exp_y(sx)$. Then we have the following proposition.

Proposition 2.2. *For all $x \in B(y,j_y)$,*

$$|\frac{\partial^\alpha}{\partial x^\alpha}u_i(x,y;H)| \leq C_\alpha(1+|y|)^{-|\alpha|-2i}, \tag{2.6}$$

where C_α is a constant independent of x, y.

Proof. See [27, Proposition 2.1] for the proof.

Definition 2.4. We fix a large integer N and define for $k \in \{0,1\}$ a parametrix $f_k(t,x,y;H)$ by

$$
\begin{aligned}
f_k(t,x,y;H) :=& T(t,x,y)(u_0(x,y;H) + tu_1(x,y;H) + \cdots \\
&+ t^{n-k}u_{n-k}(x,y;H))|dy|^{\frac{1}{2}} \\
&+ \rho(t)T(t,x,y)(t^{n-k+1}u_{n-k+1}(x,y;H) + \cdots \\
&+ t^{n+N}u_{n+N}(x,y;H))|dy|^{\frac{1}{2}},
\end{aligned} \tag{2.7}
$$

$$
T(t,x,y) := (4\pi t)^{-n} \exp\{-\frac{d(x,y)^2}{4t}\}, \tag{2.8}
$$

where ρ is a cut off function defined by $\rho(t) = 1$ on $[0,1]$, $\rho(t) = 0$ on $[2,\infty)$ and $|\frac{d\rho}{dt}| \leq 2$. Set

$$
F_k(t,x,y;H) := K^E(t,x,y;H) - f_k(t,x,y;H). \tag{2.9}
$$

From [2, Theorem 2.26], we obtain the following proposition.

Proposition 2.3.

$$
\begin{aligned}
&(\frac{\partial}{\partial t} + H_x)F_k\ (t,x,y;H) \\
&\qquad = (4\pi)^{-n}t^N e^{-\frac{d(x,y)^2}{4t}} B_x u_{n+N} \quad (0 < t \leq 1)
\end{aligned} \tag{2.10}
$$

and

$$
\begin{aligned}
(\frac{\partial}{\partial t} + H_x)F_k(t,x,y;H) =& (4\pi)^{-n}e^{-\frac{d(x,y)^2}{4t}}t^{-k}B_x u_{n-k} \\
&+ (\frac{\partial}{\partial t} + H_x)\{\rho \cdot T(t^{n-k+1}u_{n-k+1} + \cdots \\
&+ t^{n+N}u_{n+N})\} \quad (t \geq 1).
\end{aligned} \tag{2.11}
$$

Introducing

$$
G_k(t,x,y;H) := \chi_y(x)(\frac{\partial}{\partial t} + H_x)F_k(t,x,y;H)
$$

and

$$
H_k(t,x,y;H) := \int_0^t d\tau \int_X K^E(t-\tau,x,z;H)G_k(\tau,z,y;H)dv,
$$

yields the following lemma.

Lemma 2.3. *On the domain* $t \le 1 + |y|^2$,

$$\sup_{[0,t]\times B(y,\frac{1}{2}j_y)} |F_k(\cdot,\cdot,y) - H_k(\cdot,\cdot,y)| \le C\Big\{ \sup_{[0,t]\times \partial B(y,\frac{1}{2}j_y)} |F_k(\cdot,\cdot,y)| + \sup_{[0,t]\times \partial B(y,\frac{1}{2}j_y)} |H_k(\cdot,\cdot,y)| \Big\}$$

where $C > 0$ *is a constant independent of* t, y. *Here we omit* H, E, *etc. for simplicity.*

Proof. By the Weitzenböck formula, we have for every $s \in C^\infty(\mathbb{R}_+ \times X, E)$

$$(\Delta^- - \frac{\partial}{\partial t})|s|^2 = 2|\nabla s|^2 + 2 < Fs, s > -2 < (H + \frac{\partial}{\partial t})s, s >,$$

where $\Delta^- := -d^*d$, and hence the differential inequality on $[0,\infty) \times B(y, \frac{1}{2}j_y)$

$$(\Delta^- - \frac{\partial}{\partial t})|F_k - H_k|^2 \ge \frac{-c}{1+|y|^2}|F_k - H_k|^2.$$

where $c \ge 0$ is a constant independent of t, y. This implies the following inequality:

$$(\Delta^- - \frac{\partial}{\partial t}) \exp(-ct/(1+|y|^2))|F_k - H_k|^2 \ge 0.$$

Using this inequality and applying the maximum principle to $\exp(-ct/1+|y|^2)|F_k - H_k|^2$, we obtain

$$\begin{aligned} &\sup_{[0,t]\times B(y,\frac{1}{2}j_y)} \exp(-ct/(1+|y|^2))|F_k(\cdot,\cdot,y) - H_k(\cdot,\cdot,y)| \\ &\le \sup_{[0,t]\times \partial B(y,\frac{1}{2}j_y)} \exp(-ct/(1+|y|^2))|F_k(\cdot,\cdot,y) - H_k(\cdot,\cdot,y)| \\ &\le \sup_{[0,t]\times \partial B(y,\frac{1}{2}j_y)} \exp(-ct/(1+|y|^2))|F_k(\cdot,\cdot,y)| \\ &\quad + \sup_{[0,t]\times \partial B(y,\frac{1}{2}j_y)} \exp(-ct/(1+|y|^2))|H_k(\cdot,\cdot,y)|. \end{aligned} \tag{2.12}$$

Since we restrict ourselves to the time interval $[0, 1+|y|^2]$, (2.12) gives the desired inequality.

With these preliminaries, we can state the following theorem.

Theorem 2.1. *Let (E, h, ∇^E) be an asymptotically flat vector bundle, and $F \in C^\infty(X, \mathrm{End}(E))$ be an asymptotically flat potential. Denote by $K^E(t, x, y; H)$ the heat kernel of the Schrödinger operator $H = \Delta^E + F$. Then the following estimate holds:*

$$|F_k(t, y, y; H)| \leq \begin{cases} Ct^{N+1}(1 + |y|^2)^{-(n+N+1)} & (t \leq 1), \\ C(1 + |y|^2)^{-(n-k+1)}t^{1-k} & (1 \leq t \leq 1 + |y|^2), \end{cases}$$

where C is a constant independent of t, y.

Proof. By Proposition 2.1, 2.2 and Lemma 2.3, we can prove the theorem in the same way as [27, Theorem 2.1].

3. The Hardy and Sobolev inequalities on cones

Let (X, g) be a cone; i.e., $X = \mathbb{R}_+ \times N$ and $g = dr^2 + r^2 g_N$ where (N, g_N) is a compact Riemannian manifold, $\mathbb{R}_+$ is the set of positive real number, and r is the standard coordinate of $\mathbb{R}$. We denote by $C(N)$ the cone spaned by (N, g_N), and also by $A_0^p(X)$ the space of p-forms on X with compact support. Then, we have the Hardy inequality on cones:

Proposition 3.1. *Let $(X, g) = C(N)$ be a cone of dimension $m = 2n$, and $p \neq n-1, n, n+1$. Then for every $f \in A_0^p(X)$, the following inequality holds:*

$$3\|r^{-1}f\|_{L^2}^2 \leq \|df\|_{L^2}^2 + \|\delta f\|_{L^2}^2 \tag{3.1}$$

where $\delta = - * d*$ is the adjoint of d.

Proof. See Appendix (Proposition A.1) for the proof.

Proposition 3.2. *Let (X, g) and p be the same as above. Then for every $f \in A_0^p(X)$,*

$$\|f\|_{L^{\frac{2n}{n-1}}} \leq C(\|df\|_{L^2} + \|\delta f\|_{L^2}). \tag{3.2}$$

Proof. Since the Sobolev inequality holds on cones, we have

$$\|f\|_{L^{\frac{2n}{n-1}}}^2 \leq C\|d|f|\|_{L^2}^2.$$

From Kato's inequality $|d|f|| \leq |\nabla f|$ (cf.[19]) where ∇ is the connection on $\wedge^p TX$ induced by the Levi-Civita connection, it follows that

$$\|f\|^2_{L^{\frac{2n}{n-1}}} \leq C(\Delta^B f, f)$$

where $\Delta^B := \nabla^*\nabla$ is the Bochner Laplacian. Thus the Weitzenböck formula (cf.[2]) leads to

$$\|f\|^2_{L^{\frac{2n}{n-1}}} \leq C\{(\Delta^H f, f) + (Rf, f)\},$$

where $\Delta^H := (d+\delta)^2$ is the Hodge Laplacian, and $R := \Delta^B - \Delta^H$ is a 0-th order differential operator. Since $|R| \leq Cr^{-2}$, we obtain the desired inequality from Proposition 3.1.

We now consider the Kähler case. Let (X, g) be a cone as before. We assume that (X, g) is a Kähler manifold. In this case we say that (X, g) is a conical Kähler manifold. Although there is no Hardy inequality for $(n-1), n, (n+1)$ forms on cones, we can have the inequality for $(0, n-1)$- and $(0, n)$-forms in the Kähler case under a certain curvature condition. To state the condition, we prepare some notations. Let the Ricci curvature and the scalar curvature be denoted by Ric_X and σ_X respectively.

If we express $g = \sum_{i\bar{j}} g_{i\bar{j}} dz^i d\bar{z}^j$ and $\mathrm{Ric}_X = \sum_{i\bar{j}} \rho_{i\bar{j}} dz^i d\bar{z}^j$, then

$$\rho_{i\bar{j}} = \partial_i \bar{\partial}_j \log(\det(g)), \quad \sigma_X = \sum_{i\bar{j}} g^{i\bar{j}} \rho_{i\bar{j}},$$

where $\partial_i = \frac{\partial}{\partial z^i}$ and $\det(g) := \det(g_{i\bar{j}})$. Since Ric_X is a quadratic form on TX, we can define $\lambda(x)$ by the largest eigenvalue of Ric_X on $T_x X$. Then we define $\lambda_+(x) := \max\{\lambda(x), 0\}$.

Theorem 3.1. *Let (X, g) be a conical Kähler manifold such that*

$$\inf_X r^2(\sigma_X(x) - \lambda_+(x)) > -(n-1)^2. \tag{3.3}$$

Then for every $0 \leq q \leq n$, Hardy's and Sobolev's inequalities hold; i.e., for every $f \in A_0^{0,q}(X)$,

$$\|r^{-1} f\|^2_{L^2} \leq C(\|\bar{\partial} f\|^2_{L^2} + \|\bar{\partial}^* f\|^2_{L^2}) \tag{3.4}$$

and

$$\|f\|^2_{L^{\frac{2n}{n-1}}} \leq C\|\bar{\partial} f\|^2_{L^2} + \|\bar{\partial}^* f\|^2_{L^2}). \tag{3.5}$$

Since the Kähler condition of (X, g) implies $2(\|\bar{\partial} f\|_{L^2}^2 + \|\bar{\partial}^* f\|_{L^2}^2) = \|df\|_{L^2} + \|\delta f\|_{L^2}$, the theorem holds for $q < n-1$ by Propositions 3.1 and 3.2. For the proof in the remaining cases ($q = n-1, n$), we need more some lemmas and propositions.

Lemma 3.1. *If* (3.3) *holds, there is a constant* $\epsilon > 0$ *such that*

$$-\operatorname{Ric}_X + \sigma_X g_X \geq \{\epsilon - (n-1)^2\} r^{-2} g_X \tag{3.6}$$

and

$$\sigma_X \geq \{\epsilon - (n-1)^2\} r^{-2}. \tag{3.7}$$

Proof. First we remark that the curvature tensor R_X satisfies $|R_X(rx)| = r^{-2}|R_X(x)|$ since (X, g) is a cone. From the condition (3.3) there is a constant $\epsilon > 0$ such that for every $x \in X$

$$r^2(\sigma_X(x) - \lambda_+(x)) \geq \epsilon - (n-1)^2. \tag{3.8}$$

Since $\lambda_+ \geq 0$, (3.7) follows immediately. By the definition we have

$$\operatorname{Ric}_X(\xi, \xi) \leq \lambda(x) g_X(\xi, \xi) \leq \lambda_+(x) g_X(\xi, \xi)$$

for every $\xi \in T_x X$. Substituting the above inequality in (3.8) gives

$$r^2\{-\operatorname{Ric}_X(\xi, \xi) + \sigma_X(x) g_X(\xi, \xi)\} \geq \{\epsilon - (n-1)^2\} g_X(\xi, \xi),$$

and obtain (3.6).

Proposition 3.3. *Let* K_X *be the canonical bundle of* X, *and denote the Hermitian connection of* K_X *by* ∇_{K_X}. *Then, for every* $f \in A_0^{n,0}(X) = C_0^\infty(K_X)$,

$$2\|\bar{\partial} f\|_{L^2}^2 = \|\nabla_{K_X} f\|_{L^2}^2 + (\sigma_X f, f). \tag{3.9}$$

Here we consider f *as an element of* $A_0^{n,0}(X)$ *and* $C_0^\infty(K_X)$ *on the left-hand and the right-hand sides of* (3.9) *respectively.*

Proof. Let $\nabla_{K_X} = \nabla_{1,0} + \nabla_{0,1}$ be the decomposition of the connection into holomorphic and anti-holomorphic part. Then $\bar{\partial} = \nabla_{0,1}$. Define $\Box$ and $\bar{\Box}$ by

$$\Box := \nabla_{0,1}^* \nabla_{0,1}, \quad \bar{\Box} := \nabla_{1,0}^* \nabla_{1,0}.$$

Then by the Nakano formula (cf.[21]), we have

$$\square - \bar{\square} = \sqrt{-1}[e(R_{K_X}), \Lambda], \tag{3.10}$$

where $e(R_{K_X})$ is the exterior multiplication by R_{K_X}, the curvature of K_X, Λ is the interior multiplication by the Kähler form, and $[a,b]$ is defined by $[a,b] := a \circ b - b \circ a$.

We denote by Δ^B the Bochner Laplacian defined by $\Delta^B := \nabla^*_{K_X}\nabla_{K_X}$. Then, using the Kähler identities (cf.[21]), we obtain

$$\Delta^B = \square + \bar{\square}. \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$2\|\bar{\partial} f\|^2_{L^2} = \|\nabla_{K_X} f\|^2_{L^2} - (\sqrt{-1}\Lambda e(R_{K_X})f, f). \tag{3.12}$$

Since $\sqrt{-1}\Lambda e(R_{K_X}) = -\sigma_X$ in the Kähler case, we have (3.9) from (3.12).

Let TX be denoted by the holomorphic tangent bundle of X, and Ω_X^{n-1} by the holomorphic vector bundle $\wedge^{n-1}TX^*$. Then there is a canonical identification

$$\begin{aligned} i : K_X \otimes TX \ni dz^1 \cdots dz^n \otimes \frac{\partial}{\partial z^j} \\ \to (-1)^{j-1} dz^1 \cdots dz^{j-1} \wedge dz^{j+1} \cdots dz^n \in \Omega_X^{n-1}. \end{aligned} \tag{3.13}$$

which preserves Hemitian metrics and connections.

Proposition 3.4. *Let $\nabla_{K_X \otimes TX}$ be the Hermitian connection on $K_X \otimes TX$. Then,*

$$2\|\bar{\partial} f\|^2_{L^2} = \|\nabla_{K_X\otimes TX} f\|^2_{L^2} - \mathrm{Ric}_X(f, \bar{f}) + (\sigma_X f, f), \tag{3.14}$$

where $f \in A_0^{n-1,0}(X)$ on the left-hand side and $f \in C_0^\infty(K_X \otimes TX)$ on the right-hand side. Here $\mathrm{Ric}_X(f,\bar{f})$ is defined by

$$\mathrm{Ric}_X(f, \bar{f}) := \int_X \sum_{i,j} \rho_{i\bar{j}} f_i \bar{f}_j dv_X, \tag{3.15}$$

where $\mathrm{Ric}_X = \sum_{ij} \rho_{i\bar{j}} dz^i d\bar{z}^j$ and $f = \sum_i f_i dz^1 \cdots dz^n \otimes \frac{\partial}{\partial z^i}$ in holomorphic normal coordinates.

Proof. Let $\nabla_{K_X \otimes TX} = \nabla_{1,0} + \nabla_{0,1}$, $\Box$, $\bar{\Box}$ and Δ^B be the same as in the previous proposition. Since i preserves the metric and connections,

$$(3.16) \qquad |\bar{\partial} f| = |\nabla_{0,1} f|,$$

where $f \in A_0^{n-1,0}(X)$ on the left-hand side and $f \in C_0^\infty(K_X \otimes TX)$ on the right-hand side. As before by the Nakano formula

$$(3.17) \qquad \Box - \bar{\Box} = \sqrt{-1}[e(R_{K_X \otimes TX}), \Lambda]$$

where $R_{K_X \otimes TX}$ is the curvature tensor of $K_X \otimes TX$. Also the Kähler identity gives

$$(3.18) \qquad \Delta^B = \Box + \bar{\Box}.$$

Combining (3.16), (3.17) and (3.18) yields

$$(3.19) \qquad 2\|\bar{\partial} f\|_{L^2}^2 = \|\nabla_{K_X \otimes TX} f\|_{L^2}^2 - (\sqrt{-1}\Lambda e(R_{K_X \otimes TX}) f, f).$$

Since $e(R_{K_X \otimes TX}) = e(R_{TX}) + e(K_X)$, $\sqrt{-1}\Lambda e(R_{K_X}) = -\sigma_X$ and

$$(3.20) \qquad < \sqrt{-1}\Lambda e(R_{TX}) f, f > = \sum_{ij} \rho_{i\bar{j}} f_i \bar{f}_j$$

in holomorphic normal coordinates in the Kähler case, we have the desired equality from (3.19) and (3.20).

Lemma 3.2. *Let (X, g) be a cone $C(N)$, and E be a Hermitian vector bundle with a metric compatible connection ∇^E on X. Then the following inequality holds:*

$$(3.21) \qquad \int_{C(N)} |\nabla^E f|^2 dv_X \geq (n-1)^2 \int_{C(N)} r^{-2} f^2 dv_X$$

for every $f \in C_0^\infty(E)$.

Proof. Since $T\mathbb{R}_+$ is perpendicular to TN, we have $|\nabla^E f| \geq |\nabla^E_{\frac{\partial}{\partial r}} f|$, and therefore

$$(3.22) \qquad \begin{aligned} \int_{C(N)} |\nabla^E f|^2 dv_X &\geq \int_N dv_N \int_0^\infty r^{2n-1} |\nabla^E_{\frac{\partial}{\partial r}} f|^2 dr \\ &\geq \int_N dv_N \int_0^\infty r^{2n-1} (\frac{\partial}{\partial r}|f|)^2 dr, \end{aligned}$$

where we have used Kato's inequality $|\nabla^E_{\frac{\partial}{\partial r}} f| \geq |\frac{\partial}{\partial r}|f||$. From (3.22) and the Hardy inequality ($\lambda = 2n-1$ in this case)

$$\int_0^\infty r^\lambda (\frac{d}{dr}g)^2 \geq \frac{(\lambda-1)^2}{4}\int_0^\infty r^{\lambda-2} f^2 dr, \tag{3.23}$$

it follows that

$$\int_{C(N)} |\nabla^E f|^2 dv_X \geq (n-1)^2 \int_{C(N)} r^{-2} f^2 dv_X.$$

Proof of Theorem 3.1. First we prove (3.4). When $q = n$, by Lemma 3.1, (3.7), Proposition 3.3 and Lemma 3.2, for every $f \in A_0^{0,n}(X)$, we obtain

$$\begin{aligned} 2\{\|\bar\partial f\|^2_{L^2} + \|\bar\partial^* f\|^2_{L^2}\} &= 2\|\bar\partial \bar f\|^2_{L^2} \\ &= \|\nabla_{K_X} \bar f\|^2_{L^2} + (\sigma_X \bar f, \bar f) \\ &\geq \|\nabla_{K_X} \bar f\|^2_{L^2} + (\{\epsilon - (n-1)^2\} r^{-2} \bar f, \bar f) \\ &\geq \epsilon \|r^{-1} f\|^2_{L^2}. \end{aligned} \tag{3.24}$$

When $q = n-1$, from Lemma 3.1, (3.6), Proposition 3.4 and Lemma 3.2, we have for every $g \in A_0^{0,n-1}(X)$,

$$\begin{aligned} 2\{\|\bar\partial g\|^2_{L^2} + \|\bar\partial^* g\|^2_{L^2}\} &= 2\|\bar\partial \bar g\|^2_{L^2} \\ &= \|\nabla_{K_X \otimes TX} \bar g\|^2_{L^2} - \mathrm{Ric}_X(\bar g, g) + (\sigma_X \bar g, \bar g) \\ &\geq \|\nabla_{K_X \otimes TX} \bar g\|^2_{L^2} + (\{\epsilon - (n-1)^2\} \bar g, \bar g) \\ &\geq \epsilon \|r^{-1} g\|^2_{L^2}. \end{aligned} \tag{3.25}$$

This completes the proof of the Hardy inequality. Next we prove (3.5).

When $q = n$, the Weitzenböck formula and the Hardy inequality yields

$$\begin{aligned} \|f\|^2_{L^{\frac{2n}{n-1}}} &\leq C\|\nabla_{K_X} \bar f\|^2_{L^2} \\ &\leq C\{2\|\bar\partial \bar f\|^2_{L^2} - C'' \|r^{-1} \bar f\|^2_{L^2}\} \\ &\leq C'\|\bar\partial \bar f\|^2_{L^2} \\ &= C'\{\|\bar\partial f\|^2_{L^2} + \|\bar\partial^* f\|^2_{L^2}\}. \end{aligned} \tag{3.26}$$

The case $q = n-1$ can be proven in the same way and is left to the reader.

Corollary 3.1. *Under the same assumption of Theorem 3.1, the following inequality holds:*

$$\|\nabla f\|_{L^2}^2 \leq C(\|\bar{\partial} f\|_{L^2}^2 + \|\bar{\partial}^* f\|_{L^2}^2) \tag{3.27}$$

for every $f \in A_0^{0,q}(X)$.

Proof. From the Weitzenböck formula, we have

$$\|\nabla f\|_{L^2}^2 \leq \|\bar{\partial} f\|_{L^2}^2 + C\|r^{-1} f\|_{L^2}^2, \tag{3.28}$$

which together with Theorem 3.1 gives our desired inequality.

Let $\pi : \tilde{X} \to X$ be a desingularization of X, and $\tilde{g}$ be a Kähler metric on $\tilde{X}$ such that $\tilde{g} = \pi^* g$ on $X' := \tilde{X} - \pi^{-1}(C_{0,1}(N))$. Setting

$$\mathcal{H}_q(\tilde{X}) := \{f \in \Omega_q(\tilde{X}); \int_{\tilde{X}} \frac{|f|^2}{1+r^2} dv < \infty, \int_{\tilde{X}} |f|^{\frac{2n}{n-1}} dv < \infty\}, \tag{3.29}$$

where $\Omega_q(\tilde{X})$ is the space of holomorphic q-forms on $\tilde{X}$, we have the following theorem.

Theorem 3.2. *Let X satisfy the same conditions as in Theorem 3.1. If $\mathcal{H}_q(\tilde{X}) = 0$, then the following inequalities hold:*

$$\|(1+r)^{-1} f\|_{L^2}^2 \leq C(\|\bar{\partial} f\|_{L^2}^2 + \|\bar{\partial}^* f\|_{L^2}^2), \tag{3.30}$$

$$\|f\|_{L^{\frac{2n}{n-1}}}^2 \leq C(\|\bar{\partial} f\|_{L^2}^2 + \|\bar{\partial}^* f\|_{L^2}^2), \tag{3.31}$$

$$(\Delta_q f, f) \leq C(\Box_{0,q} f, f) \tag{3.32}$$

for every $f \in A^{0,q}(\tilde{X})$, where $\Delta_q := \nabla^* \nabla$ is the Bochner Laplacian on q-forms, and $\Box_{0,q} := (\bar{\partial} + \bar{\partial}^*)^2$ is the Hodge Laplacian on $(0,q)$-forms on $\tilde{X}$.

Proof. Since $\tilde{X}$ is Kähler, (3.31) and (3.32) follow from (3.30) in the same way as Theorem 3.1 and Corollary 3.1. Therefore we need only to prove (3.30). By using a partition of unity from Theorem 3.1, we have

$$\|(1+r)^{-1} f\|_{L^2}^2 \leq C(\|\bar{\partial} f\|_{L^2}^2 + \|\bar{\partial}^* f\|_{L^2}^2 + \|f\|_{L^2(K)}^2) \tag{3.33}$$

and

$$\|f\|_{L^{\frac{2n}{n-1}}}^2 \leq C(\|\bar{\partial} f\|_{L^2}^2 + \|\bar{\partial}^* f\|_{L^2}^2 + \|f\|_{L^2(K)}^2), \tag{3.34}$$

where $K := \pi^{-1}(C_{0,1}(N))$. If there is no constant such that

$$\|f\|^2_{L^2(K)} \leq C(\|\bar{\partial} f\|^2_{L^2} + \|\bar{\partial}^* f\|^2_{L^2}), \tag{3.35}$$

then there is a sequence $\{f_n\}$ such that

$$\|f_n\|^2_{L^2(K)} = 1, \quad \|\bar{\partial} f_n\|^2_{L^2} + \|\bar{\partial}^* f_n\|^2_{L^2} \to 0.$$

By (3.34) we obtain $\|(1+r)^{-1} f_n\|_{L^2} + \|f_n\|^2_{L^{\frac{2n}{n-1}}} \leq M$ for some $M < \infty$. Taking a subsequence and using the Rellich lemma, we can find an element g such that

$$\|g\|_{L^2(K)} = 1, \quad \|(1+r)^{-1} g\|_{L^2} + \|g\|^2_{L^{\frac{2n}{n-1}}} \leq M \tag{3.36}$$

and

$$\bar{\partial} g = 0, \quad \bar{\partial}^* g = 0. \tag{3.37}$$

Since $\tilde{X}$ is Kähler, we know $\bar{g} \in \mathcal{H}_q(\tilde{X})$. From (3.36) $\bar{g}$ is a nonzero element of $\mathcal{H}_q(\tilde{X})$. This contradicts $\mathcal{H}_q(X) = 0$. Therefore there is a constant C satisfying (3.35). Combining (3.33) and (3.35) gives the desired inequality.

4. Conic degeneration of Kähler manifolds and behavior of the heat kernels

Let $\{(\tilde{M}, g_\epsilon)\}$ be the conic degenerating family of Kähler manifolds considered in section 1. Throughout this section we assume that (X, g) satisfies the same condition as Theorem 3.2; i.e.,

$$\inf_X r^2(\sigma_X(x) - \lambda_+(x)) > -(n-1)^2 \quad \textit{and} \quad \mathcal{H}_q(\tilde{X}) = 0.$$

We denote by $\Box^\epsilon_{0,q}$ the Hodge Laplacian $(\bar{\partial} + \bar{\partial}^*)^2$ on (0,q)-forms on $\tilde{M}$ with respect to g_ϵ. Let $K^\epsilon_{0,q}(t, x, y)$ be the heat kernel of $\Box^\epsilon_{0,q}$. Then its trace

$$\text{Tr } e^{-t\Box^\epsilon_{0,q}} = \int_{\tilde{M}} \text{tr } K^\epsilon_{0,q}(t, x, x) dv_\epsilon(x)$$

has the following asymptotic expansion as $t \to 0$.

$$\text{Tr } e^{-t\Box^\epsilon_{0,q}} \sim (4\pi t)^{-\frac{m}{2}} \{a_0(\epsilon, q) + t a_1(\epsilon, q) + t^2 a_2(\epsilon, q) + \cdots\} \tag{4.1}$$

where $a_i(\epsilon, q)$ is computed by using the parametrix constructed in section 2 as follows. Let $u_i(x; \Box^{\epsilon}_{0,q}) := u_i(x, x; \Box^{\epsilon}_{0,q})$ be the heat kernel invariant constructed in the same way as (2.4) and (2.5) for $H = \Box^{\epsilon}_{0,q}$. Then $a_i(\epsilon, q)$ is given by

$$(4.2)\qquad \begin{aligned} a_i(\epsilon, q) &= \int_{\tilde{M}} \text{tr } u_i(x; \Box^{\epsilon}_{0,q}) dv_{\epsilon} \\ &= \int_{\tilde{U}} \text{tr } u_i(x; \Box^{\epsilon}_{0,q}) dv + \int_{\tilde{M}-\tilde{U}} \text{tr } u_i(x; \Box^{\epsilon}_{0,q}) dv_{\epsilon} \\ &= \epsilon^{m-2i} \int_{B(o,\epsilon^{-1})} \text{tr } u_i(x, q) dv + \int_{\tilde{M}-\tilde{U}} \text{tr } u_i(x; \Box^{\epsilon}_{0,q}) dv_{\epsilon}, \end{aligned}$$

where $u_i(x, q) := u_i(x, x; \Box_{0,q})$ is the same one as (2.4) and (2.5) for $(\tilde{X}, g_{\tilde{X}})$ and $\tilde{U} = \pi^{-1}(C^*_{0,1}(N))$. Our goal in this section is the following theorem.

Theorem 4.1. *For $(\tilde{X}, g_{\tilde{X}})$ define $b_{0,q}(\epsilon, t)$ by*

$$(4.3)\qquad b_{0,q}(\epsilon, t) \log \frac{1}{t} := \int_{\tilde{B}(\epsilon^{-1})-\tilde{B}(\epsilon^{-1}\sqrt{t})} (4\pi)^{-n} \text{ tr } u_n(x, q) dv.$$

where $\tilde{B}(r) := \pi^{-1}(C^*_{0,r}(N))$. Then the following estimate holds for $t \in (0, 1]$

$$(4.4)\qquad |\operatorname{Tr} e^{-t\Box^{\epsilon}_{0,q}} - (4\pi t)^{-n} \sum_{i=0}^{n-1} a_i(\epsilon, q) t^i - b_{0,q}(\epsilon, t) \log \frac{1}{t}| \leq C,$$

where $C > 0$ is a constant independent of ϵ, t.

For the proof of the theorem, we need two lemmas.

Lemma 4.1. *Let $L^{\epsilon}_{0,q}(t, x, y)$ be the heat kernel of the Hodge Laplacian on $(0, q)$-froms on $\tilde{M}$ with respect to the metric $\epsilon^{-2} g_{\epsilon}$. Let $K_{0,q}(t, x, y)$ be the heat kernel of the Hodge Laplacian on $(0, q)$-forms on $\tilde{X}$ with respect to $g_{\tilde{X}}$. Then, under the identification between $\tilde{M}$ and $\tilde{X}$ on $\tilde{B}(\epsilon^{-1})$, the following estimate holds:*

(4.5)

$$|L^{\epsilon}_{0,q}(t, y, y) - K_{0,q}(t, y, y)| \leq \begin{cases} \dfrac{C t^{N+1}}{(1 + |y|^2)^{n+N+1}}, & (t \leq 1) \\ \dfrac{Ct}{(1 + |y|^2)^{n+1}}, & (1 \leq t \leq 1 + |y|^2) \end{cases}$$

for $y \in \tilde{B}(\epsilon^{-1})$, where $|\cdot|$ is the operator norm on $\operatorname{End}(\wedge^q T^{"*} \tilde{X})$, *and N is a fixed large integer.*

Proof. Let $\Box_{0,q}$ be the Hodge Laplacian on $(0,q)$-forms on $\tilde{X}$. Since both $L^{\epsilon}_{0,q}(t)$ and $K_{0,q}(t)$ satisfy the same heat equation on $B(y,j_y)$, in the same way as that used in the proof of Lemma 2.3 we find

$$\begin{aligned}(4.6)\quad &\sup_{[0,t]\times B(y,\frac{1}{2}j_y)} |L^{\epsilon}_{0,q}(\cdot,\cdot,y) - K_{0,q}(\cdot,\cdot,y)| \\ &\leq C\{\sup_{[0,t]\times\partial B(y,\frac{1}{2}j_y)} |L^{\epsilon}_{0,q}(\cdot,\cdot,y)| + \sup_{[0,t]\times\partial B(y,\frac{1}{2}j_y)} |K_{0,q}(\cdot,\cdot,y)|\}\end{aligned}$$

for $y \in \tilde{B}(\epsilon^{-1})$ where $C > 0$ is a constant independent of t, y. Applying Lemma 2.1 to both $L^{\epsilon}_{0,q}(t)$ and $K_{0,q}(t)$ gives

$$\begin{aligned}(4.7)\quad &\sup_{[0,t]\times B(y,\frac{1}{2}j_y)} |L^{\epsilon}_{0,q}(\cdot,\cdot,y) - K_{0,q}(\cdot,\cdot,y)| \\ &\leq C\sup_{[0,t]} t^{-n}\exp(-\frac{\gamma(1+|y|^2)^2}{t}) \\ &\leq C_k(1+|y|^2)^{-n}\sup_{[0,t]}(t(1+|y|^2)^{-1})^k.\end{aligned}$$

Setting $k = N+1$ for $t \leq 1$ and $k = 1$ for $1 \leq t \leq 1+|y|^2$ in (4.7), we have the desired estimate.

Lemma 4.2. *There is a constant $C > 0$ independent of ϵ, t such that for $t \in (0,1]$*

$$(4.8)\qquad |K^{\epsilon}_{0,q}(t,x,x)| \leq Ct^{-n}.$$

Proof. Using a partition of unity, we have from Theorem 3.2 it follows that

$$(4.9)\qquad (\Delta^{\epsilon}_q f, f)_{\epsilon} \leq C((\Box^{\epsilon}_{0,q} + I)f, f)_{\epsilon}$$

for every $f \in A^{0,q}(\tilde{M})$ where $(\cdot,\cdot)_{\epsilon}$ implies the inner product with respect to g_{ϵ}. Note that the inequalities in Theorem 3.2 are scaling-invariant. Then, by the definition of the heat operator,

$$(4.10)\qquad (\exp(-t\Delta^{\epsilon}_q)f, f)_{\epsilon} \geq Ce^{-Ct}(\exp(-Ct\Box^{\epsilon}_{0,q})f, f)_{\epsilon}.$$

Let $H^{\epsilon}_q(t,x,y)$ be the heat kernel of Δ^{ϵ}_q. Then from (4.10) we have

$$(4.11)\qquad |K^{\epsilon}_{0,q}(t,x,x)| \leq Ce^{Ct}|H^{\epsilon}_q(t,x,x)|,$$

which together with the theorem of [19] yields

$$|H_q^\epsilon(t,x,x)| \le K_{0,0}^\epsilon(t,x,x). \tag{4.12}$$

By [27, Theorem 3.1] thus we obtain

$$K_{0,0}^\epsilon(t,x,x) \le Ct^{-n}. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) our desired estimate.

Proof of Theorem 4.1. We compute as follows:

(4.14)

$$\begin{aligned}
I := &\Big| \int_{\bar{M}} \mathrm{tr}\, K_{0,q}^\epsilon(t,x,x) dv_\epsilon(x) \\
&-(4\pi t)^{-n} \sum_{i=0}^{n-1} a_i(\epsilon,q)t^i - b_{0,q}(\epsilon,t)\log\frac{1}{t} \Big| \\
\le &\Big| \int_{\bar{U}} \Big\{ \mathrm{tr}\, K_{0,q}^\epsilon(t,x,x) - (4\pi t)^{-n} \sum_{i=0}^{n-1} \mathrm{tr}\, u_i(x;\Box_{0,q}^\epsilon)t^i \Big\} dv_\epsilon(x) \\
&-b_{0,q}(\epsilon,t)\log\frac{1}{t} \Big| \\
&+\Big| \int_{M'} \Big\{ \mathrm{tr}\, K_{0,q}^\epsilon(t,x,x) - (4\pi t)^{-n} \sum_{i=0}^{n-1} \mathrm{tr}\, u_i(x;\Box_{0,q}^\epsilon)t^i \Big\} dv_\epsilon(x) \Big| \\
= &\Big| \int_{\bar{B}(\epsilon^{-1})} \Big\{ \mathrm{tr}\, L_{0,q}^\epsilon(\frac{t}{\epsilon^2},x,x) - (\frac{\epsilon^2}{4\pi t})^n \sum_{i=0}^{n-1} \mathrm{tr}\, u_i(x,q)(\frac{t}{\epsilon^2})^i \Big\} dv \\
&- \int_{\bar{B}(\epsilon^{-1})-\bar{B}(\epsilon^{-1}\sqrt{t})} (4\pi)^{-n}\, \mathrm{tr}\, u_n(x,q) dv \Big| + O(1) \\
\le &\int_{\bar{B}(\epsilon^{-1})-\bar{B}(\epsilon^{-1}\sqrt{t})} |F_0(\epsilon^{-2}t,x,x)| dv \\
&+ \int_{\bar{B}(\epsilon^{-1})-\bar{B}(\epsilon^{-1}\sqrt{t})} |L_{0,q}^\epsilon(\epsilon^{-2}t,x,x) - K_{0,q}(\epsilon^{-2}t,x,x)| dv \\
&+ \int_{\bar{B}(\epsilon^{-1}\sqrt{t})} |L_{0,q}^\epsilon(\epsilon^{-2}t,x,x)| dv \\
&+ \int_{\bar{B}(\epsilon^{-1}\sqrt{t})} |(4\pi\epsilon^{-2}t)^{-n} \sum_{i=0}^{n-1} \mathrm{tr}\, u_i(x,q)(\epsilon^{-2}t)^i| dv \\
&+\rho(\epsilon^{-2}t) \sum_{i=1}^{N} (\epsilon^{-2}t)^i \int_{\bar{B}(\epsilon^{-1})-\bar{B}(\epsilon^{-1}\sqrt{t})} |\mathrm{tr}\, u_{n+i}(x,q)| dv \\
&+O(1).
\end{aligned}$$

where we have used the formula

$$K_{0,q}^\epsilon(t,x,y) = \epsilon^{-2n} L_{0,q}^\epsilon(\epsilon^{-2}t,x,y). \tag{4.15}$$

We set $\delta_i(t,\epsilon)$ for the i-th term of the right-hand side of the last inequality of (4.14), and shall estimate each $\delta_i(t,\epsilon)$. Since $1+|x|^2 \le \epsilon^{-2}t$ for $x \in \tilde{B}(\epsilon^{-1}) - \tilde{B}(\epsilon^{-1}\sqrt{t})$, from Theorem 2.1 for $k=0$ it follows that

$$\begin{aligned}\delta_1(t,\epsilon) &\le \int_{\epsilon^{-1}\sqrt{t}}^{\epsilon^{-1}} C\epsilon^{-2}t\frac{r^{2n-1}}{(1+r^2)^{n+1}}dr \\ &\le C\epsilon^{-2}t\int_{\epsilon^{-1}\sqrt{t}}^{\infty} r^{-3}dr \\ &= \frac{1}{2}C.\end{aligned} \tag{4.16}$$

Similarly using Lemma 4.1 we obtain

$$\delta_2(t,\epsilon) \le C. \tag{4.17}$$

Moreover, from (4.15),

$$\int_{\tilde{B}(\epsilon^{-1}\sqrt{t})} |L^{\epsilon}_{0,q}(\epsilon^{-2}t,x,x)|dv = \int_{\tilde{B}(\sqrt{t})} |K^{\epsilon}_{0,q}(t,x,x)|dv_\epsilon, \tag{4.18}$$

which together with Lemma 4.2 gives

$$\delta_3(t,\epsilon) \le Ct^{-n}\operatorname{vol}(B(\sqrt{t})) \le C'. \tag{4.19}$$

Thus by Proposition 2.2, we have

$$\begin{aligned}\delta_4(t,\epsilon) &\le \int_0^{\epsilon^{-1}\sqrt{t}} C\sum_{i=0}^{n-1}(\epsilon^{-2}t)^{-n+i}\frac{r^{2n-1}}{(1+r^2)^i}dr \\ &= C\sum_{i=0}^{n-1}\frac{1}{n-i}\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\delta_5(t,\epsilon) &\le \rho(\epsilon^{-2}t)C\sum_{i=1}^{N}(\epsilon^{-2}t)^i\int_0^{\epsilon^{-1}}\frac{r^{2n-1}}{(1+r^2)^{n+i}}dr \\ &\le 2^N C\sum_{i=1}^{N}\int_0^{\infty}\frac{r^{2n-1}}{(1+r^2)^{n+i}}dr,\end{aligned} \tag{4.21}$$

since $\rho(\epsilon^{-2}t)=0$ when $t \ge 2\epsilon^2$. Combining (4.16), (4.17), (4.19), (4.20) and (4.21) hence yields our desired inequality.

5. Conic degeneration of Kähler manifolds and the first eigenvalue of Laplacians

Let $\{(\tilde{M}, g_\epsilon)\}$ be the same as in the previous section, and Y be the exceptional divisor of $\tilde{X}$; i.e., $Y := \pi^{-1}(p)$. In this section, we assume that Y is smooth and satisfies $H^0(Y, \Omega_Y^q) = 0$ for $0 < q < n$. Let $\lambda_i^q(\epsilon) > 0$ be the nonzero eigenvalue of $\Box_{0,q}^\epsilon$. Then we have the following proposition.

Proposition 5.1. *There is a constant $C_q > 0$ such that for every $0 \leq q \leq n$ and $\epsilon \in \Delta^*$,*

$$\lambda_1^q(\epsilon) \geq C_q. \tag{5.1}$$

Proof. Noting that the Sobolev inequality is scaling-invariant, from Theorem 3.2 we have

$$\|s\|_{L^{\frac{2n}{n-1}},\epsilon} \leq C(\|\bar{\partial}s\|_{L^2,\epsilon} + \|\bar{\partial}^* s\|_{L^2,\epsilon} + \|s\|_{L^2,\epsilon}). \tag{5.2}$$

We assume $\lambda_1^q(\epsilon) \to 0$ as $\epsilon \to 0$. Let $s_1(\epsilon)$ be the normalized eigenform for $\lambda_1^q(\epsilon)$; i.e.,

$$\Box_{0,q}^\epsilon s_1(\epsilon) = \lambda_1(\epsilon) s_1(\epsilon), \quad \|s_1(\epsilon)\|_{L^2,\epsilon} = 1. \tag{5.3}$$

Then we show that

$$s_1(\epsilon) \to s \quad L_1^2(\tilde{M}, g) \tag{5.4}$$

for some nonzero s such that $\bar{s}$ is holomorphic and $\bar{s} \perp H^0(\tilde{M}, \Omega^q)$ in the inner product of (M, g) where $L_1^2(\tilde{M}, g)$ is the completion of $A_0^{0,q}(M)$ by the norm $\|f\|_{L^2} + \|df\|_{L^2} + \|\delta f\|_{L^2}$. Let $\{f_1, \cdots, f_N\}$ be a fixed basis of $H^0(\tilde{M}, \Omega^q)$. Then $\bar{s}_1(\epsilon) \perp \{f_1, \cdots, f_N\}$ in the inner product of g_ϵ; i.e.,

$$\int_{\tilde{M}} \bar{s}_1(\epsilon) \wedge \bar{f}_i \wedge \omega_\epsilon^{n-q} = 0. \tag{5.5}$$

We set

$$\tilde{s}_1(\epsilon) := \rho_\epsilon \cdot \bar{s}_1(\epsilon), \tag{5.6}$$

where

$$\rho_\epsilon(r) = \begin{cases} 1 & (r \geq \sqrt{|\epsilon|}), \\ \frac{-2}{\log|\epsilon|^{2\alpha}} \int_{|\epsilon}^{r} \frac{dr}{r} & (|\epsilon| \leq r \leq \sqrt{|\epsilon|}), \\ 0 & (r \leq \epsilon). \end{cases} \tag{5.7}$$

For simplicity, we consider the case $\alpha = 1$. Then

$$\begin{aligned} \|\tilde{s}_1(\epsilon)\|^2_{L^2_1,0} &= \|\tilde{s}_1(\epsilon)\|^2_{L^2_1,\epsilon} \\ &\leq \|\tilde{s}_1(\epsilon)\|^2_{L^2,\epsilon} + \|\bar{\partial}\tilde{s}_1(\epsilon)\|^2_{L^2,\epsilon} \\ &\leq 1 + \|\bar{\partial}\tilde{s}_1(\epsilon)\|^2_{L^2,\epsilon}. \end{aligned} \tag{5.8}$$

Since $\bar{\partial}\tilde{s}_1(\epsilon) = \rho_\epsilon \bar{\partial}\bar{s}_1(\epsilon) + \bar{\partial}\rho_\epsilon \wedge \bar{s}_1(\epsilon)$, we have

$$\begin{aligned} &\|\bar{\partial}\tilde{s}_1(\epsilon)\|^2_{L^2,\epsilon} \\ &\leq 2(\lambda_1^2(\epsilon) + \|\bar{\partial}\rho_\epsilon \wedge \bar{s}_1(\epsilon)\|^2_{L^2,\epsilon}) \\ &\leq 2\lambda_1^2(\epsilon) + \frac{8}{(\log|\epsilon|)^2} \int_{|\epsilon|}^{\sqrt{|\epsilon|}} \frac{|s_1(\epsilon)|^2}{r^2} dv_X \\ &\leq 2\lambda_1^2(\epsilon) + \frac{8}{(\log|\epsilon|)^2} \left\{ \int_N dv_N \int_{|\epsilon|}^{\sqrt{|\epsilon|}} (r^{-2})^n r^{2n-1} dr \right\}^{\frac{1}{n}} \\ &\qquad \times \left\{ \int_N \int_{|\epsilon|}^{\sqrt{|\epsilon|}} |s_1(\epsilon)|^{\frac{2n}{n-1}} dv_X \right\}^{\frac{n-1}{n}} \\ &\leq 2\lambda_1^2(\epsilon) + 2\frac{\mathrm{vol}(N)}{(\log|\epsilon|)^2} \|s_1(\epsilon)\|^2_{L^{\frac{2n}{n-1}},\epsilon}. \end{aligned} \tag{5.9}$$

By (5.2) we know $\|s_1(\epsilon)\|_{L^{\frac{2n}{n-1}},\epsilon} \leq C$. Therefore from (5.9) it follows that $\|\tilde{s}_1(\epsilon)\|_{L^2_1,0}$ is uniformly bounded by a constant C independent of ϵ. Then due to the Rellich lemma, there is an element $s \in L^2_1(\tilde{M}, g_0)$ such that $\tilde{s}_1(\epsilon)$ converges to s weakly in $L^2_1(\tilde{M}, g_0)$ and strongly in $L^2(\tilde{M}, g_0)$, taking a suitable subsequence. Since

$$\begin{aligned} \|\tilde{s}_1(\epsilon)\|^2_{L^2,0} &\geq \|s_1(\epsilon)\|^2_{L^2,\epsilon} - \|s_1(\epsilon)\|^2_{L^2(C_{0,\sqrt{|\epsilon|}}(N))} \\ &= 1 - \|s_1(\epsilon)\|^2_{L^2(C_{0,\sqrt{|\epsilon|}}(N))} \\ &\geq 1 - \mathrm{vol}(C_{0,\sqrt{|\epsilon|}}(N))^{\frac{1}{2n}} \|s_1(\epsilon)\|_{L^{\frac{2n}{n-1}},\epsilon} \\ &\geq 1 - C\sqrt{|\epsilon|}, \end{aligned} \tag{5.10}$$

we obtain $\|s\|_{L^2,0} = 1$. In particular $s \neq 0$. From (5.9), we have $\bar{\partial} s = 0$. Since s is square integrable with respect to $g = g_0$, we may regard s as a meromorphic q-form on $\tilde{M}$, which may possibly have a pole along Y and is holomorphic outside of it. This implies that for some $m \geq 0$

$$(5.11) \qquad s \in H^0(\tilde{M}, \Omega^q(m[Y])),$$

where $[Y]$ is the line bundle defined by the divisor Y.

Case 1 $(1 < q < n)$. There is a coordinate system $\{U_\alpha\}$ near Y such that on U_α, s can be represented by

$$s = a_\alpha \wedge d\zeta_\alpha + b_\alpha \ ,$$

where $Y \cap U_\alpha = \{\zeta_\alpha = 0\}$, $a_\alpha \in \Gamma(U_\alpha - Y, \Omega^{q-1})$ and $b_\alpha \in \Gamma(U_\alpha - Y, \Omega^q)$. Since the bundle $[Y]|_Y$ is defined by the system $\{g_{\alpha\beta} := \zeta_\alpha \zeta_\beta^{-1}|_Y\}$, we have on $U_\alpha \cap U_\beta$

$$(5.12) \qquad a_\beta = a_\alpha g_{\alpha\beta}, \quad b_\beta = a_\alpha \zeta_\beta + b_\alpha \ .$$

Assume that a_α has a pole of order k along Y. Then

$$a_\alpha = \zeta_\alpha^{-k} a'_\alpha \ ,$$

where $a'_\alpha \in \Gamma(U_\alpha, \Omega^{q-1})$, and a'_α does not vanish identically on U_α. Thus we can regard the system $\{a'_\alpha|_Y\}$ as an element of $H^0(Y, \Omega^{q-1}([Y]^{k-1}|_Y))$. If $k > 1$, since $[Y]|_Y$ is negative on Y, we have

$$H^0(Y, \Omega^{q-1}([Y]^{k-1}|_Y)) = 0.$$

This implies $a'_\alpha = 0$ identically and is a contradiction. If $k = 1$, then $\{a'_\alpha|_Y\} \in H^0(Y, \Omega^{q-1}) = 0$, and $a'_\alpha = 0$ identically on U_α, a contradiction. Therefore $k \geq 0$ and a_α is holomorphic on U_α. From (5.12), by the same argument as on $\{a_\alpha\}$, we know that b_α is regular along Y. Thus

$$(5.13) \qquad s \in H^0(\tilde{M}, \Omega^q),$$

which implies that $s = 0$, since $\overline{s_1(\epsilon)}$ is perpendicular to $H^0(\tilde{M}, \Omega^q)$. This contradicts the condition $\|s\|_{L^2,0} = 1$. Hence we obtain a uniform lower bound

$$(5.14) \qquad \lambda_1^q(\epsilon) \geq C > 0.$$

Case 2 (q=0,1 or n). We first prove the case $q = 0$. In this case, since $X - p$ is connected, we know the uniform lower bound of the first eigenvalue from [27, Proposition 4.1]. Since $\lambda_1^1(\epsilon)$ is greater than either $\lambda_1^0(\epsilon)$ or $\lambda_1^2(\epsilon)$ (cf. [17, Lemma 1.6.5]), we obtain the uniform lower bound of the first eigenvalue for $q = 1$. When $q = n$, since $\lambda_1^n(\epsilon) \geq \lambda_1^{n-1}(\epsilon)$ we also have the uniform lower bound.

Proposition 5.2. *There is a constant $C > 0$ independent of ϵ such that*

$$\lambda_l^q(\epsilon) \geq C l^{\frac{2}{m}}. \tag{5.15}$$

Proof. By Theorem 4.1,

$$\mathrm{Tr}\, \exp(-t\Box_{0,q}^{\epsilon}) = \sum_{i=0}^{\infty} e^{-t\lambda_i^q(\epsilon)} \leq C t^{-n}$$

for $0 < t \leq 1$ where C is a constant independent of ϵ. Setting $t = \lambda_l^q(\epsilon)^{-1} C_q \leq 1$ where C_q is the same constant as in (5.1), we have

$$e^{-C_q} l \leq \sum_{i=0}^{l} e^{-\frac{C_q \lambda_i^q(\epsilon)}{\lambda_l^q(\epsilon)}} \leq C \lambda_l^q(\epsilon)^n. \tag{5.16}$$

Inequality (5.15) follows immediately from (5.16).

6. Proof of the Main Theorem

In this section, we use the same notation as in sections 4 and 5. Let $\zeta_{0,q}(s,\epsilon)$ be the spectral zeta function of $\Box_{0,q}^{\epsilon}$, and

$$\mathrm{Tr}\, e^{-t\Box_{0,q}^{\epsilon}} \sim (4\pi t)^{-n}\{a_0(\epsilon,q) + a_1 t(\epsilon,q) + \cdots\}$$

be the asymptotic expansion of trace of the heat kernel as $t \to 0$. Then, by the definition, we have the following expression of $\zeta_{0,q}(s,\epsilon)$:

$$
(6.1)\quad
\begin{aligned}
\zeta_{0,q}(s,\epsilon) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - h_q)dt \\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\{\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - (4\pi t)^{-n}\sum_{i=0}^{n} a_i(\epsilon,q)t^i\}dt \\
&\quad + \frac{(4\pi)^{-n}}{\Gamma(s)}\sum_{i=0}^{n}\frac{a_i(\epsilon,q)}{s-n+i} - \frac{h_q}{s\Gamma(s)} \\
&\quad + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}(\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - h_q)dt,
\end{aligned}
$$

where $h_q = \dim H^q(\tilde{M}, \mathcal{O}_{\tilde{M}})$. Since $\zeta_{0,q}(0,\epsilon) = (4\pi)^{-n}a_n(\epsilon,q)$, we have the following formula:

$$
(6.2)\quad
\begin{aligned}
\frac{d}{ds}|_{s=0}\zeta_{0,q}(s,\epsilon) &= \int_0^1 \{\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - (4\pi t)^{-n}\sum_{i=0}^{n} a_i(\epsilon,q)t^i\}\frac{dt}{t} \\
&\quad + (4\pi)^{-n}\sum_{i=0}^{n-1}\frac{a_i(\epsilon,q)}{n-i} - (4\pi)^{-n}\Gamma'(1)a_n(\epsilon,q) \\
&\quad + \Gamma'(1)h_q + \int_1^\infty t^{-1}(\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - h_q)dt.
\end{aligned}
$$

Using the above formula in our conic degenerating case, we obtain the following formula.

Proposition 6.1. *As* $\epsilon \to 0$,

(6.3)

$$
\begin{aligned}
\frac{d}{ds}|_{s=0}\zeta(s,\epsilon) &= -\int_{\epsilon^2}^1 \frac{dt}{t}\int_{B(|\epsilon|^{-1}\sqrt{t})}(4\pi)^{-n}\,\mathrm{tr}\ u_n(x,q)dv + O(\log\frac{1}{|\epsilon|}) \\
&= -\int_1^{\epsilon^{-2}} \frac{dt}{t}\int_{B(\sqrt{t})}(4\pi)^{-n}\,\mathrm{tr}\ u_n(x,q)dv + O(\log\frac{1}{|\epsilon|}),
\end{aligned}
$$

where $u_n(x,q) = u_n(x,x;\Box_{0,q})$ *is the same as* (2.4).

Proof. We set the following I_1, I_2, I_3 and compute each of them:

$$
\begin{aligned}
I_1 &:= \int_0^1 \left\{\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - (4\pi t)^{-n}\sum_{i=0}^{n} a_i(\epsilon,q)t^i\right\}\frac{dt}{t}, \\
I_2 &:= (4\pi)^{-n}\sum_{i=0}^{n-1}\frac{a_i(\epsilon,q)}{n-i} - (4\pi)^{-n}\Gamma'(1)a_n(\epsilon,q) + \Gamma'(1)h_q, \\
I_3 &:= \int_1^\infty t^{-1}(\mathrm{Tr}\ e^{-t\Box_{0,q}^{\epsilon}} - h_q)dt.
\end{aligned}
$$

First we compute:
(6.4)

$$
\begin{aligned}
I_1 &= \int_0^{|\epsilon|^2} \{\mathrm{Tr}\ e^{-t\Box^\epsilon_{0,q}} - (4\pi t)^{-n} \sum_{i=0}^{n} a_i(\epsilon, q) t^i\} \frac{dt}{t} \\
&\quad + \int_{|\epsilon|^2}^{1} \{\mathrm{Tr}\ e^{-t\Box^\epsilon_{0,q}} - (4\pi t)^{-n} \sum_{i=0}^{n} a_i(\epsilon, q) t^i\} \frac{dt}{t} \\
&= O(\int_0^{|\epsilon|^2} \frac{dt}{t} |\epsilon|^{-2} t) \\
&\quad + \int_{|\epsilon|^2}^{1} \{\mathrm{Tr}\ e^{-t\Box^\epsilon_{0,q}} - (4\pi t)^{-n} \sum_{i=0}^{n-1} a_i(\epsilon, q) t^i - b_{0,q}(\epsilon, t) \log \frac{1}{t}\} \frac{dt}{t} \\
&\quad - \int_{|\epsilon|^2}^{1} \frac{dt}{t} \int_{B(|\epsilon|^{-1}\sqrt{t})} (4\pi)^{-n}\ \mathrm{tr}\ u_n(x, q) dv \\
&\quad + \int_{|\epsilon|^2}^{1} \int_{M-\bar{U}_\epsilon} \mathrm{tr}\ u_n(x; \Box^\epsilon_{0,q}) dv_\epsilon \\
&= - \int_{|\epsilon|^2}^{1} \frac{dt}{t} \int_{B(|\epsilon|^{-1}\sqrt{t})} (4\pi)^{-n}\ \mathrm{tr}\ u_n(x, q) dv + O(1) + O(\int_{|\epsilon|^2}^{1} \frac{dt}{t}) \\
&= - \int_{1}^{|\epsilon|^{-2}} \frac{dt}{t} \int_{B(\sqrt{t})} (4\pi)^{-n}\ \mathrm{tr}\ u_n(x, q) dv + O(\log \frac{1}{|\epsilon|}).
\end{aligned}
$$

Since $a_i(\epsilon, q)$ is bounded for $i < n$ and is of logarithmic order by Proposition 2.2, we have

$$I_2 = O(\log \frac{1}{|\epsilon|}). \tag{6.5}$$

Finally, we estimate I_3. Let $\{0 = \lambda_1^q(\epsilon) < \lambda_2^q(\epsilon) \leq \cdots\}$ be the eigenvalue of $\Box^\epsilon_{0,q}$, counted with multiplicities. Then, from Proposition 5.2 it follows that

$$
\begin{aligned}
\mathrm{Tr}\ e^{-t\Box^\epsilon_{0,q}} &= \sum_{i=1}^{\infty} e^{-t\lambda_i^q(\epsilon)} \\
&\leq \sum_{i=1}^{k} e^{-t\lambda_i^q(\epsilon)} + Ct^{-n} \\
&\leq k e^{-t\lambda_1^q(\epsilon)} + Ct^{-n}.
\end{aligned} \tag{6.6}
$$

By Proposition 5.1,

$$\lambda_1^q(\epsilon) \geq C, \tag{6.7}$$

which together with (6.3), (6.6) gives

$$I_3 = O(1). \tag{6.8}$$

Combining (6.4), (6.5) and (6.8) we obtain the desired estimate.

For the proof of Theorem 0.1, we need the following result of Cheeger (cf. [13, Theorem 2.1] and [30, §5,6,7]).

Proposition 6.2. *Let $Z = C_{0,1}(N) \cup Y$ be a Kähler manifold with a conical singularity, and denote the polar coordinate of $C(N)$ by $x = (r, \omega)$. Then, the asymptotic expansion of the diagonal of the heat kernel of $\square_{0,q}$ the Friedrichs extension of the Laplacian on $(0,q)$-forms, is given by*

$$K_{0,q}(t,x,x) \sim (4\pi t)^{-n} \sum_{i=0}^{\infty} c_i(x,q) t^i, \tag{6.9}$$

where $c_i(\cdot, q) \in \mathrm{End}(\wedge^{0,q})$, and $\mathrm{tr}\ c_i(x,q) = a_i(\omega, q) r^{-2i}$ on $C_{0,1}(N)$ with some smooth function $a_i(\cdot, q)$ on N. Therefore $a_i(\omega,q)r^{-2i}$ is the heat kernel invariant, being a polynomial of derivatives of the curvature tensor of $C_{0,1}(N)$. Moreover, the asymptotic expansion of the trace of the heat kernel is given by

$$\begin{aligned} \mathrm{Tr}\ e^{-t\square_{0,q}} &= (4\pi t)^{-n} \sum_{i=0}^{n} \Big(\int_Z \mathrm{tr}\ c_i(x,q) dv_Z\Big) t^i \\ &\quad + \frac{1}{2} \int_N (4\pi)^{-n} a_n(\omega, q) dv_N \log \frac{1}{t} + O(1), \end{aligned} \tag{6.10}$$

and

$$\mathrm{Res}_{s=0}\, \zeta(s) = \frac{1}{2} (4\pi)^{-n} \int_N a_n(\omega, q) dv_N. \tag{6.11}$$

Combing Propositions 6.1 and 6.2,

Theorem 6.1. *The following asymptotic formula holds as $\epsilon \to 0$:*

$$\frac{d}{ds}\Big|_{s=0} \zeta_{0,q}(s,\epsilon) = -2\, \mathrm{Res}_{s=0}\, \zeta_{0,q}(s,0) (\log \frac{1}{|\epsilon|})^2 + O(\log \frac{1}{|\epsilon|}). \tag{6.12}$$

Proof of Theorem 0.1. Let $\mathcal{M}$ be the family of Kähler manifolds considered in section 1. Then each fiber is biholomorphic to $\tilde{M}$ in a natural way. We fix a basis of each cohomology group $H^q(\tilde{M}, \mathcal{O}_{\tilde{M}})$ and obtain an element of $\wedge^{\max} H^q(\tilde{M}, \mathcal{O}_{\tilde{M}})$, which is denoted by σ_q. Then we can define an element of the determinant of the cohomology; i.e., $\otimes_{q=0}^n \wedge^{\max} (H^q(\tilde{M}, \mathcal{O}_{\tilde{M}}))^{(-1)^q}$ by

$$\sigma := \otimes_q \sigma_q^{(-1)^q}. \tag{6.13}$$

Since $\mathcal{M} = \tilde{M} \times \Delta^*$, σ can be regarded as a holomorphic section of the Knudsen-Mumford determinant λ^{KM} on Δ^*. We also denote this holomorphic section by σ. Since each fiber $\tilde{M} \times \{\epsilon\}$ carries a Kähler metric g_ϵ, we can define the norm of $\sigma(\epsilon)$ by using the L^2-norm of each element of $H^q(\tilde{M}, \mathcal{O}_{\tilde{M}})$, and denote it by $\|\sigma(\epsilon)\|_{L^2}$. Then $\|\sigma(\cdot)\|_{L^2}$ is a smooth function on Δ^*. Since the zero set of σ is an analytic subset of $\tilde{M}$, we have the following estimate:

$$0 < C_1 \leq \|\sigma(\epsilon)\|_{L^2} \leq C_2 < \infty. \tag{6.14}$$

Following Ray-Singer (cf. [26]), we define the analytic torsion of $(\tilde{M}, g_\epsilon)$ by

$$T(\tilde{M}, g_\epsilon) := \exp(-\sum_{q=0}^{n} (-1)^q q \frac{d}{ds} \zeta_{0,q}(s, \epsilon)|_{s=0}). \tag{6.15}$$

Then the Quillen metric $\|\cdot\|_Q$ of λ^{KM} is given by

$$\|\sigma(\epsilon)\|_Q := T(\tilde{M}, g_\epsilon) \|\sigma(\epsilon)\|_{L^2}. \tag{6.16}$$

Using the familly index theorem of Bismut-Gillet-Soulé (cf. [11, Theorem 1.9]), we obtain on Δ^*

$$\partial\bar{\partial} \log \|\sigma\|_Q = 2\pi i \left[\int_{\mathcal{M}/\Delta} \mathrm{Td}(R(T\tilde{M}, G)) \right]^{(1,1)}. \tag{6.17}$$

From Proposition 1.1, the right-hand side of (6.17) extends smoothly on Δ. By Theorem 6.1, $\log \|\sigma\|_Q$ is an integrable function on Δ. Therefore as an equation of distributions on Δ, we have

$$\begin{aligned} \partial\bar{\partial} \log \|\sigma\|_Q &= 2\pi i \left[\int_{\mathcal{M}/\Delta} \mathrm{Td}(R(T\tilde{M}, G)) \right]^{(1,1)} \\ &\quad + \sum_{k=0}^{N} \sum_{a+b=k} A_{ab} \frac{d^k}{d\epsilon^a d\bar{\epsilon}^b} \delta_0 d\epsilon d\bar{\epsilon}\ , \end{aligned} \tag{6.18}$$

where A_{ab} is a constant and δ_0 is the Dirac delta function supported at the origin. From Theorem 6.1 and (6.14), it follows that $A_{ab} = 0$ for $a + b > 0$. Therefore

$$\sum_{q=0}^{n}(-1)^q q \frac{d}{ds}\zeta_{0,q}(s,\epsilon)|_{s=0} = B \log \frac{1}{|\epsilon|} + O(1), \tag{6.19}$$

where B is a constant. Thus again by Theorem 6.1, we know

$$\sum_{q=0}^{n}(-1)^q q \cdot \mathrm{Res}_{s=0}\, \zeta_{0,q}(s,0) = 0. \tag{6.20}$$

Proposition 6.3. *Let X be a Stein reduction of a negative line bundle L over a compact Kähler munifold Y. Let g_X be a conical Kähler metric of X for which* (0.6) *holds, and $g_{\tilde{X}}$ be a Kähler metric on $L = \tilde{X}$ such that $g_{\tilde{X}} = \pi^* g_X$ on $\tilde{X} - K$, where $p : \tilde{X} \to X$ is a desingularization, and K is a compact set containing the exceptional divisor Z_Y, the zero section of L. Then we have the following vanishing theorem of cohomology groups: For $0 \le q \le n$,*

$$\mathcal{H}_q(\tilde{X}) = 0. \tag{6.21}$$

Proof. We divide the proof into the case $q < n - 1$, $q = n - 1$ and $q = n$.

Case 1 (0≤q<n−1). Let f be an element of $\mathcal{H}_q(\tilde{X})$. Then $\int_{C_{1,\infty}(N)} r^{-2}|p^* f|^2 dv_X < \infty$, where N is the unit circle bundle of L, and $X = C(N)^*$. Moreover, we have

$$\begin{aligned}
&\int_{C_{0,1}(N)} r^{-2}|(p^{-1})^* f|^2 dv_X \\
&\quad = C_q \int_{C_{0,1}(N)} r^{-2}(p^{-1})^* f \wedge \overline{(p^{-1})^* f} \wedge \omega_X^{n-q} \\
&\quad = C_q \int_{p^{-1}(C_{0,1}(N))} r^{-2} f \wedge \bar{f} \wedge ((p^{-1})^* \omega_X)^{n-q} \\
&\quad \le C_q \sup_V |f|^2 \int_V r^{-2} \omega_{\tilde{X}}^q \wedge ((p^{-1})^* \omega_X)^{n-q},
\end{aligned} \tag{6.22}$$

where C_q is a constant which depends only on q. In the above computation, we have used the convention that $V = p^{-1}(C_{0,1}(N))$ and that ω_X,

$\omega_{\tilde{X}}$ are Kähler forms of g_X, $g_{\tilde{X}}$ respectively. If g_Y is a Kähler metric on Y, then we have the following relation.

$$\begin{aligned} &(p^{-1})^* g_X \sim |\zeta|^{2(a-1)}(|\zeta|^2 g_Y + |d\zeta|^2), \\ &g_{\tilde{X}} \sim g_Y + |d\zeta|^2, \\ &r \sim |\zeta|^a, \end{aligned} \tag{6.23}$$

where $g_1 \sim g_2$ implies that g_1 is quasi-isometric to g_2, and r has the same order as $|\zeta|^a$. From (6.23) it follows that

$$\begin{aligned} \int_V r^{-2}\omega_{\tilde{X}}^q \wedge ((p^{-1})^*\omega_X)^{n-q} &\leq C\int_V r^{-2}\omega_{\tilde{X}}^{n-2} \wedge ((p^{-1})^*\omega_X)^2 \\ &\leq C\int_V |\zeta|^{-2a}(\omega_Y + d\zeta \wedge d\bar{\zeta})^{n-2} \\ &\qquad \wedge |\zeta|^{4(a-1)}(|\zeta|^2\omega_Y + d\zeta \wedge d\bar{\zeta})^2 \\ &\leq C\int_V |\zeta|^{2(a-1)}\omega_Y^{n-1} \wedge d\zeta \wedge d\bar{\zeta} \\ &\leq C\int_\Delta |\zeta|^{2(a-1)} d\zeta \wedge d\bar{\zeta} < \infty, \end{aligned} \tag{6.24}$$

where ω_Y is the Kähler form of g_Y. Therefore, if $f \in \mathcal{H}_q(\tilde{X})$, we have

$$\int_X r^{-2}|(p^{-1})^* f|^2 dv_X < \infty. \tag{6.25}$$

Let ρ_ϵ be the cut-off function defined by (5.7), setting $a = 1$. Let $f_\epsilon := \rho_\epsilon (p^{-1})^* f$. Then the Hardy inequality gives

$$\|r^{-1} f_\epsilon\|_{L^2}^2 \leq C(\|df_\epsilon\|_{L^2}^2 + \|\delta f_\epsilon\|_{L^2}^2). \tag{6.26}$$

Since $df_\epsilon = d\rho_\epsilon \wedge (p^{-1})^* f + \rho_\epsilon d(p^{-1})^* f$, we estimate each term of the right-hand side. By the definition of ρ_ϵ, we have

$$\|d\rho_\epsilon \wedge (p^{-1})^* f\|_{L^2}^2 \leq (\log \epsilon)^{-2}\|r^{-1}(p^{-1})^* f\|_{L^2}^2. \tag{6.27}$$

Since f satisfies $\bar{\partial} f = 0$ and $\|f\|_{L^{\frac{2n}{n-1}}} < \infty$, we can apply integratation by parts to f and obtain $df = \delta f = 0$. Therefore

$$\|df_\epsilon\|_{L^2} \leq |\log \epsilon|^{-1}\|r^{-1} f\|_{L^2}. \tag{6.28}$$

In the same way, we have

$$\|\delta f_\epsilon\|_{L^2} \leq |\log \epsilon|^{-1}\|r^{-1} f\|_{L^2}. \tag{6.29}$$

From (6.28) and (6.29) it follows that

$$r^{-1}(p^{-1})^*f = \lim_{\epsilon\to 0} r^{-1}f_\epsilon = 0, \tag{6.30}$$

which implies $f = 0$ and completes the proof for this case.

Case 2 (q=n). Let f be an element of $\mathcal{H}_n(\tilde{X})$. Since $K_{\tilde{X}}|_Y = K_Y - L$ from the adjunction formula, $(K_{\tilde{X}} - (a-1)\pi^*L)|_Y = K_Y - aL$ where we identify Y and Z_Y. Since $H^0(Y, K_Y - aL) = 0$ by the assumption, $H^0(Y, K_Y - mL) = 0$ for $m \le a$. This implies that $(\zeta^{-m}f)|_{Z_Y} = 0$ for $m \le a-1$ where ζ is the local defining function of Z_Y. Therefore we have the following estimate near Z_Y:

$$|f| \le C|\zeta|^a. \tag{6.31}$$

Hence,

$$\begin{aligned}\int_{C_{0,1}(N)} r^{-2}|(p^{-1})^*f|^2 dv_X &= C_n \int_{p^{-1}(C_{0,1}(N))} r^{-2} f\wedge\bar{f} \\ &\le C_n \int_{p^{-1}(C_{0,1}(N))} d\zeta\wedge d\bar{\zeta}\wedge\omega_Y^{n-1} \\ &< \infty,\end{aligned} \tag{6.32}$$

since r has the same order as $|\zeta|^a$. Therefore,

$$\int_X r^{-2}|(p^{-1})^*f|^2 dv_X < \infty. \tag{6.33}$$

Thus in the same way as the Case 1, we can show $f = 0$ and completes the proof for this case.

Case 3 (q=n-1). Let f be an element of $\mathcal{H}_{n-1}(\tilde{X})$. Since $H^0(Y, K_Y) = 0$, f vanishes along Z_Y. We can express $f\wedge\bar{f}$ as follows.

$$f\wedge\bar{f} = a\omega_Y^{n-1} + b\omega_Y^{n-2}\wedge d\zeta\wedge d\bar{\zeta}. \tag{6.34}$$

Since f vanishes along Z_Y, we have $|a(\zeta,\cdot)| \le C|\zeta|$. Since ω_X is quasi-isometric to $|\zeta|^{2(a-1)}(d\zeta\wedge d\bar{\zeta} + \omega_Y)$,

$$f\wedge\bar{f}\wedge\omega_X \le C|\zeta|^{2(a-1)}(|a| + |b|\cdot|\zeta|)d\zeta\wedge d\bar{\zeta}\wedge\omega_Y^{n-1}. \tag{6.35}$$

Since r has the same order as $|\zeta|^a$,

$$r^{-2}|(p^{-1})^*f|^2 dv_X \le C|\zeta|^{-1} d\zeta\wedge d\bar{\zeta}\wedge\omega_Y^{n-1}, \tag{6.36}$$

which implies that

$$\int_X r^{-2}|(p^{-1})^* f|^2 dv_X < \infty. \tag{6.37}$$

Therefore, as before, we can show $f = 0$.

Proof of the Main Theorem. By Theorem 0.1 and Proposition 6.3, it is sufficient to show that $H^0(Y, \Omega_Y^q) = 0$ for $0 \leq q < n$. From the condition $K_Y - aL < 0$, we know $K_Y < 0$, which implies $H^0(Y, \Omega_Y^q) = 0$ for $0 \leq q < n$ by the Kodaira vanishing theorem.

7. Spectral zeta function of 2-dimensional Kähler manifolds

In the 2-dimensional case, (0.7) holds without the assumptions in the statement of the Main Theorem.

Theorem 7.1. *Let (M, g) be a compact Kähler surface with a conical singularity p. Then the following equality holds:*

$$\sum_{q=0}^{2} (-1)^q q \cdot \mathrm{Res}_{s=0}\, \zeta_{0,q}(s) = 0. \tag{7.1}$$

Since $\sum_{q=0}^{2} (-1)^q \zeta_{0,q}(s)$ vanishes identically, we have the following corollary.

Corollary 7.1.

$$\mathrm{Res}_{s=0}\, \zeta_{0,0}(s) = \frac{1}{2} \mathrm{Res}_{s=0}\, \zeta_{0,1}(s) = \mathrm{Res}_{s=0}\, \zeta_{0,2}(s). \tag{7.2}$$

Proof. We can express (M, g) by $M = C_{0,1}^*(N) \cup M'$ where M' is a manifold with boundary N. Let $K_{0,q}(t, x, y)$ be the heat kernel of $\Box_{0,q}$ on $C(N)$. By the conformal homogeneity of $C(N)$, we have

$$\begin{aligned} &\mathrm{tr}\ K(t, x, x) dv = f(r; \omega, t) r^{2n-1} dr \wedge dv_N, \\ &f(r, \omega, t) = r^{-2n} f(1, \omega, r^{-2} t) \end{aligned} \tag{7.3}$$

where ω is the variable on N. As $t \to 0$, the following asymptotic expansion holds:

$$f(r, \omega, t) \sim (4\pi t)^{-n} \sum_{i \geq 0} \alpha_i(r, \omega) t^i, \quad \alpha_i(r, \omega) = r^{-2i} \alpha_i(1, \omega). \tag{7.4}$$

By Cheeger's computation, we obtain

$$\text{Res}_{s=0}\,\zeta_{0,q}(s) = \frac{(4\pi)^{-n}}{2}\int_N \alpha_n(\omega)dv_N\ , \tag{7.5}$$

where $\alpha_i(\omega) := \alpha_i(1,\omega)$. Set $M_\epsilon := C_{\epsilon,1}(N) \cup M'$, and let $K^\epsilon_{0,q}(t,x,y)$ be the heat kernel of M_ϵ with the Dirichlet boundary condition. Then [24, Lemma 5.3] for $x \in C_{2\epsilon,1}(N)$

$$K_{0,q}(t,x,x) = K^\epsilon_{0,q}(t,x,x) + O(e^{-\frac{\delta}{t}}). \tag{7.6}$$

In the interior of M_ϵ, we have the following asymptotic expansion:

$$K^\epsilon_{0,q}(t,x,x) \sim (4\pi t)^{-n}\sum_{i=0}^{\infty} a_i(x,q)t^i, \tag{7.7}$$

where a_i is the usual heat kernel invariant. From (7.7) it follows that on $C_{2\epsilon,1}(N)$,

$$a_i(x,q) = \alpha_i(r,\omega) = r^{-2i}\alpha_i(\omega). \tag{7.8}$$

Note that the relation (7.8) is extended to the whole cone $C(N)$. Therefore

$$\int_N \alpha_2(\omega)dv_N = \frac{1}{\log r}\int_{C_{1,r}(N)} a_2(x)dv, \tag{7.9}$$

which together with (7.5) gives

$$\text{Res}_{s=0}\,\zeta_{0,q}(s) = \frac{(4\pi)^{-n}}{2}\frac{1}{\log r}\int_{C_{1,r}(N)} a_2(x)dv. \tag{7.10}$$

For the 2-dimensional Kähler manifolds, we have the following formulas (cf. [17, Lemma 4.8.17] and [1, pp.82 and 225]):

$$a_2(x,0) = \frac{1}{(4\pi)^2 360}\{5|\tau|^2 - 2|\rho|^2 + 2|R|^2\}, \tag{7.11}$$

$$a_2(x,1) = \frac{1}{(4\pi)^2 360}\{-20|\tau|^2 + 86|\rho|^2 - 11|R|^2\}, \tag{7.12}$$

$$a_2(x,2) = \frac{1}{(4\pi)^2 360}\{20|\tau|^2 - 32|\rho|^2 + 2|R|^2\}, \tag{7.13}$$

$$c_1(T'M)^2 = \frac{1}{(4\pi)^2}\{|\tau|^2 - 2|\rho|^2\}, \tag{7.14}$$

$$c_2(T'M) = \frac{1}{32(\pi)^2}\{|\tau|^2 - 4|\rho|^2 + |R|^2\}, \tag{7.15}$$

where τ is the scalar curvature, ρ is the Ricci curvature, R is the total Riemannian curvature, and $T'M$ is the holomorphic tangent bundle of M. From (7.11), (7.12), (7.13), (7.14) and (7.15), we have

$$\sum_{q=0}^{2}(-1)^q q a_2(x,q) = \frac{2}{15}c_1(T'M)^2 + \frac{1}{12}c_2(T'M). \tag{7.16}$$

Since $c_2(T'M) = \chi(TM)$ and $2c_2(T'M) - c_1(T'M)^2 = p_1(T'M)$ for 2-dimensional Kähler manifolds, where $\chi(TM)$ is the Euler form, and $p_1(T'M)$ is the first Pontrjagin form of (M,g), the right-hand side of (7.16) is expressed by a sum of $\chi(TM)$ and $p_1(T'M)$. Since both $\chi(TM)$ and $p_1(T'M)$ vanish on cones, the right-hand side of (7.16) is equal to 0. Therefore by (7.10) and (7.16), we obtain (7.1).

8. Examples

In this section, we treat some examples of Kähler manifolds with a conical singularity which satisfy the conditions in the Main Theorem.

Proposition 8.1. *Let $F \in \mathbb{C}[z_0,\cdots,z_n]$ be a homogeneous polynomial of degree m. Set*

$$X(F) = \{(z_0,\cdots,z_n) \in \mathbb{C}^{n+1}; F(z_0,\cdots,z_n) = 0\},$$

$$g_{X(F)} := |dz_0|^2 + \cdots + |dz_n|^2|_{X(F)}, \tag{8.1}$$

and denote by $\rho_{X(F)}$ the Ricci curvature and by $\sigma_{X(F)}$ the scalar curvature as before. Then the following inequalities hold:

$$-\frac{\|\nabla^2 F\|_{op}^2}{\|\nabla F\|^2} g_{X(F)} \leq \rho_{X(F)} \leq 0, \quad -n\frac{\|\nabla^2 F\|_{op}^2}{\|\nabla F\|^2} \leq \sigma_{X(F)} \leq 0, \tag{8.2}$$

where

$$\nabla F := (\frac{\partial F}{\partial z_0}, \cdots, \frac{\partial F}{\partial z_n}) \in \mathbb{C}^{n+1}, \quad \nabla^2 F := (\frac{\partial^2 F}{\partial z_i \partial z_j})_{0 \le i,j \le n} \in M(n+1; \mathbb{C}),$$

$\| \cdot \|$ *is the standard norm on* $\mathbb{C}^{n+1}$, *and* $\| \cdot \|_{op}$ *is the operator norm on* $M(n+1; \mathbb{C})$.

Proof. We identify the Hermitian metric with the Kähler form, and the Ricci curvature with the Ricci form. For simplicity, we use the following conventions:

$$F_i := \frac{\partial F}{\partial z_i}, \quad F_{ij} := \frac{\partial^2 F}{\partial z_i \partial z_j}.$$

By computation, we have

$$\rho_{X(F)} = -\sqrt{-1}\|\nabla F\|^{-4} \sum_{i,j=0}^{n} \{\|\nabla F\|^2 \delta_{ij} - F_i \bar{F}_j\} dF_i \wedge d\bar{F}_j|_{X(F)}, \tag{8.3}$$

from which it follows that

$$\rho_{X(F)} \le 0, \quad \sigma_{X(F)} \le 0, \tag{8.4}$$

and also that

$$\begin{aligned} \rho_{X(F)} &\ge -\sqrt{-1}\|\nabla F\|^{-2} \sum_{i=0}^{n} dF_i \wedge d\bar{F}_i \\ &= -\sqrt{-1}\|\nabla F\|^{-2} \sum_{j,k=0}^{n} (\sum_{i=0}^{n} F_{ij} \bar{F}_{ik}) dz_j \wedge d\bar{z}_k \\ &\ge -\|\nabla F\|^{-2} \|\nabla^2 F\|_{op}^2 \omega_{X(F)}, \end{aligned} \tag{8.5}$$

where $\omega_{X(F)} := \sqrt{-1}\sum_{i,j=0}^{n} dz_i \wedge d\bar{z}_j|_X$ is the induced Kähler form of (X, g). Moreover, by (8.5) we obtain

$$\sigma_{X(F)} \ge -n\|\nabla F\|^{-2}\|\nabla^2 F\|_{op}^2. \tag{8.6}$$

Proposition 8.2. *Let* $F \in \mathbb{C}[z_0, \cdots, z_n]$ *be a homogeneous polynomial of degree* m *such that* $Y(F) := \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n(\mathbb{C}); F(z_0, \cdots, z_n) = 0\}$ *is a smooth hypersurface in* $\mathbb{P}^n(\mathbb{C})$. *Let* L *be the* *tautological line bundle of* $\mathbb{P}^n(\mathbb{C})$. *Then* $X(F)$ *is the Stein reduction of* $L|_Y$, *and* $g_{X(F)}$ *satisfies*

$$T_\lambda^* g_{X(F)} = |\lambda|^2 g_{X(F)}. \tag{8.7}$$

Proof. Since the Stein reduction of L is $\mathbb{C}^{n+1}$, it is clear that $X(F)$ is the Stein reduction of $L|_{Y(F)}$. Since $g_{X(F)}$ is the restriction of the Euclidean metric, it is also clear that (8.7) holds.

Definition 8.1. Let V be a n-dimensional complex space and p be an isilated singularity of V. p is said to be a hypersurface homogeneous singularity if there is a homogeneous polynomial $F \in \mathbb{C}[z_0, \cdots, z_n]$ and a isomorphism from a neighborhood of p to a neighborhood of $0 \in X(F)$. F is said to be the defining equation of p.

Definition 8.2. Let p be a hypersurface homogeneous singularity defined by $F \in \mathbb{C}[z_0, \cdots, z_n]$, and g be a Hermitian metric of V. g is said to be the induced Euclidean metric near p if

$$g|_{U_p} = i^*(|dz_0|^2 + \cdots + |dz_n|^2), \tag{8.8}$$

where U_p is a neighborhood of p, and i is the identification map from U_p to $X(F)$.

Theorem 8.1. *Let* (M, g) *be a compact Kähler space of pure dimension* n *with at most a hypersurface homogeneous singularity* p *defined by* $F \in \mathbb{C}[z_0, \cdots, z_n]$. *If* g *is the induced Euclidean metric near* p *and*

$$\sup_{x \in X(F)} \frac{\|x\|^2 \cdot \|\nabla^2 F(x)\|_{op}^2}{\|\nabla F(x)\|^2} < \frac{(n-1)^2}{n}, \quad m := \deg F < n, \tag{8.9}$$

then (0.7) *holds for* (M, g).

Proof. By the assumption of the theorem and Proposition 8.2, (M, g) is a Kähler manifold with a conical singularity associated to the line bundle $\pi : L|_{Y(F)} \to Y(F)$, and $(X(F), g_{X(F)})$ is the Stein reduction of $L|_{Y(F)}$. From Proposition 8.1 and (8.9), (0.6) is satisfied for $(X(F), g_{X(F)})$. Therefore if $K_{Y(F)} - L|_{Y(F)} < 0$ is satisfied, (M, g) satisfies the assumptions of the Main Theorem. Since $K_{Y(F)} = (n + 1 - m)L|_{Y(F)}$ by the adjunction formula, inequality (8.9) gives

$$K_{Y(F)} - L|_{Y(F)} = (n - m)L|_{Y(F)} < 0. \tag{8.10}$$

Corollary 8.1. *Let p be the n-dimensional node, i.e., the hypersurface homogeneous singularity defined by the following polynomial:*

$$F_2(z) = z_0^2 + z_1^2 + \cdots + z_n^2. \tag{8.11}$$

Let (M, g) be a compact Kähler space of dimension $n > 2$ with at most one node. If g is the induced Euclidean metric near the singularity, then (0.7) holds for (M, g).

Proof. By Theorem 8.1, it is sufficient to verify (8.9) for $(X(F_2), g_{X(F_2)})$. From (8.11) it follows that

$$\|x\|^2 \cdot \|\nabla^2 F_2(x)\|_{op}^2 \|\nabla F_2(x)\|^{-2} = 1$$

for every $x \in X(F_2)$. Since $n > 2$, (8.9) holds.

Example 8.1. Let $M = \{[z_0 : z_1 : \cdots : z_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C}); \sum_{i=0}^{n} z_i^2 = 0\}$ be the compactification of $X(F_2) = M \cap U_{n+1}$ in $\mathbb{P}^{n+1}(\mathbb{C})$ where $U_i := \{[z_0 : z_1 : \cdots : z_{n+1}]; z_i \neq 0\}$. Set $D_\infty := \mathbb{P}^{n+1}(\mathbb{C}) - U_{n+1}$. Since, on U_{n+1}, both the Euclidean metric g_E and the Fubini-Study metric g_{FS} have the potential function

$$\phi_E := \|z\|^2 = \sum_{i=0}^{n} |z_i|^2, \quad \phi_{FS}(z) := \log(1 + \|z\|^2), \tag{8.12}$$

we can patch g_E and g_{FS} to obtain a new Kähler metric $g^{(n+1)}$ on $\mathbb{P}^{n+1}(\mathbb{C})$ such that $g^{(n+1)} = g_E$ on $\mathbb{B}(1)$ and $g^{(n+1)} = g_{FS}$ on $\mathbb{P}^{n+1}(\mathbb{C}) - \mathbb{B}(2)$ where $\mathbb{B}(r)$ is the Euclidean ball of radius r. Set $g_M := g^{(n+1)}|_M$ to obtain a Kähler space (M, g_M). It is clear from the construction that (M, g) is an example satisfying the condition of Corollary 8.1.

Example 8.2. Let $M_m^{(n)} = \{[z_0 : z_1 : \cdots : z_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C}); F_m^{(n)}(z) := \sum_{i=0}^{n} z_i^m = 0\}$ be a compactification of $X(F_m^{(n)}) = M_m^{(n)} \cap U_{n+1}$. Let $g^{(n+1)}$ be the same Kähler metric of $\mathbb{P}^{n+1}(\mathbb{C})$ as in Example 8.1. Set $g_m^{(n+1)} := g|_{M_m^{(n+1)}}$ which coincides with the induced Euclidean metric near the singularity and with the restriction of the Fubini-Study metric near $M_m^{(n+1)} \cap D_\infty$. By computation,

$$\sup_{x \in X(F_m^{(n)})} \frac{\|x\|^2 \cdot \|\nabla^2 F_m^{(n)}(x)\|_{op}^2}{\|\nabla F_m^{(n)}(x)\|^2} = (m-1)^2 \sup_{0 \le t \le 1} \frac{1 + nt}{1 + nt^{m-1}}. \tag{8.13}$$

Set

$$\phi_n(t) := \frac{n(t-1)^2}{(n-1)^2} \{1 + \frac{1}{t-1} \{n(t-2)\}^{\frac{t-2}{t-1}}\} \tag{8.14}$$

and

$$C(n) := \sup\{m \in \mathbb{Z}_+; \phi_n(m) \leq 1\}. \tag{8.15}$$

From Proposition A.2 (see Appendix), we have

$$n(m-1)^2 \sup_{0\leq t\leq 1} \frac{1+nt}{1+nt^{m-1}} < (n-1)^2 \phi_n(m). \tag{8.16}$$

If $m \leq C(n)$, then $(M_m^{(n)}, g_m^{(n)})$ is an example satisfying the condition of Theorem 8.1. Since

$$\phi_n(t) \sim (t-1)(t-2)^{\frac{t-2}{t-1}} n^{-\frac{1}{t-1}} \leq (t-1)^2 n^{-\frac{1}{t-1}} \quad (n \to \infty),$$

there is a constant $D > 0$ independent of m and n such that if $n \geq D(m-1)^{2(m-1)}$, then $m \leq C(n)$. Therefore, there is an example satisfying the condition of Theorem 8.1 whose singularity has any arbitrary multiplicity.

Appendix

In this appendix, we prove the unequality (8.16) and the Hardy inequality on cones for p-forms.

Proposition A.1. *Let X be a cone $C(N)$ of dimension $2n$ with the conical metric $g_X = dr^2 + r^2 g_N$. Then for $p \neq n-1, n, n+1$, the following inequality holds for every $f \in A_0^p(X)$:*

$$3\|r^{-1}f\|_{L^2}^2 \leq \|df\|_{L^2}^2 + \|\delta f\|_{L^2}^2 \ , \tag{A.1}$$

where $\delta = -*d*$ is the adjoint of d.

Proof. Let $f^{(p)} = \omega_1^{(p)} + \omega_2^{(p-1)} \wedge dr$ be a p-form with compact support. By computation, we have

$$df = d_N\omega_1^{(p)} + \{d_N\omega_2^{(p-1)} - (-1)^p \frac{d}{dr}\omega_1^{(p)}\} \wedge dr \ , \tag{A.2}$$

$$\begin{aligned}\delta f = \{&r^{-2}\delta_N\omega_1^{(p)} - (-1)^p r^{-m+2(p-1)} \frac{d}{dr}(r^{m-2(p-1)}\omega_2^{(p-1)})\} \\ &-r^{-2}\delta_N\omega_2^{(p-1)} \wedge dr\end{aligned} \tag{A.3}$$

where d_N and δ_N denotes the exterior derivative and its adjoint on N respectively, and $m = 2n-1$. Again a computation leads to

$$\begin{aligned}
&\|df\|^2_{C(N)} + \|\delta f\|^2_{C(N)} \\
&= \int_0^\infty r^{m-2(p+1)}\|d_N\omega_1^{(p)}\|^2_N dr \\
&\quad + \int_0^\infty r^{m-2p}\|d_N\omega_2^{(p-1)} - (-1)^p\frac{d}{dr}\omega_1^{(p)}\|^2_N dr \\
&\quad + \int_0^\infty r^{m-2(p-1)}\|r^{-2}\delta_N\omega_1^{(p)} \\
&\quad -(-1)^p r^{-m+2(p-1)}\frac{d}{dr}\{r^{m-2(p-1)}\omega_2^{(p-1)}\}\|^2_N dr \\
&\quad + \int_0^\infty r^{m-2p}\|\delta_N\omega_2^{(p-1)}\|^2_N dr \\
&= \int_0^\infty r^{m-2(p+1)}\{\|d_N\omega_1^{(p)}\|^2_N + \|\delta_N\omega_1^{(p)}\|^2_N \\
&\quad + r^2\|d_N\omega_2^{(p-1)}\|^2_N + r^2\|\delta_N\omega_2^{(p-1)}\|^2_N \\
&\quad -2(-1)^p r(\omega_1^{(p)}, d_N\omega_2^{(p-1)})_N \\
&\quad -2(-1)^p r(\delta_N\omega_1^{(p)}, \omega_2^{(p-1)})_N\}dr \\
&\quad + \int_0^\infty r^{m-2p}\|\frac{d}{dr}\omega_1^{(p)}\|^2_N dr \\
&\quad + \int_0^\infty r^{-m+2(p-1)}\|\frac{d}{dr}(r^{m-2(p-1)}\omega_2^{(p-1)})\|^2_N dr.
\end{aligned} \tag{A.4}$$

Since $|\frac{d}{dr}\omega_1^{(p)}| \geq |\frac{d}{dr}|\omega_1^{(p)}||$, we have

$$\begin{aligned}
\int_0^\infty r^{m-2p}\|\frac{d}{dr}\omega_1^{(p)}\|^2_N dr &\geq \int_N dv_N \int_0^\infty r^{m-2p}(\frac{d}{dr}|\omega_1^{(p)}|)^2 dr \\
&\geq \frac{(m-2p-1)^2}{4}\int_0^\infty r^{m-2(p+1)}|\omega_1^{(p)}|^2 dr,
\end{aligned} \tag{A.5}$$

where we have used the following equality:

$$\begin{aligned}
\int_0^\infty r^\lambda (g'(r))^2 dr = \frac{(\lambda-1)^2}{4}\int_0^\infty r^\lambda (r^{-1}g(r))^2 dr \\
+ \int_0^\infty r\{(r^{\frac{(\lambda-1)}{2}}g(r))'\}^2 dr,
\end{aligned} \tag{A.6}$$

where $g \in C_0^\infty((0,\infty))$. Similarly

$$\begin{aligned}
\int_0^\infty r^{-m+2(p-1)}\|\frac{d}{dr}(r^{m-2(p-1)}\omega_2^{(p-1)}\|^2_N dr \\
\geq \frac{(m-2p+3)^2}{4}\int_0^\infty r^{m-2p}\|\omega_2^{(p-1)}\|^2_N dr.
\end{aligned} \tag{A.7}$$

From (A.4), (A.5) and (A.7) it follows that

$$(A.8)\qquad \begin{aligned} I :=& \int_0^\infty r^{m-2p} \|\frac{d}{dr}\omega_1^{(p)}\|_N^2 dr \\ &+ \int_0^\infty r^{-m+2(p-1)} \|\frac{d}{dr}(r^{m-2(p-1)}\omega_2^{(p-1)})\|_N^2 dr \\ \geq& \frac{(m-2p-1)^2}{4} \int_0^\infty r^{m-2(p+1)} \|\omega_1^{(p)}\|_N^2 dr \\ &+ \frac{(m-2p+3)^2}{4} \int_0^\infty r^{m-2p} \|\omega_2^{(p-1)}\|_N^2 dr. \end{aligned}$$

Using the Cauchy-Schwarz inequality, (A.4) and (A.8), we obtain

$$(A.9)\qquad \begin{aligned} &\|df\|_{C(N)}^2 + \|\delta f\|_{C(N)}^2 \\ &= \int_0^\infty r^{m-2(p+1)} dr \{ \|d_N \omega_1^{(p)}\|_N^2 + \|\delta_N \omega_1^{(p)}\|_N^2 \\ &\quad + r^2 \|d_N \omega_2^{(p-1)}\|_N^2 + r^2 \|\delta_N \omega_2^{(p-1)}\|_N^2 \\ &\quad + \frac{(m-2p-1)^2}{4} \|\omega_1^{(p)}\|_N^2 \\ &\quad + \frac{(m-2p+3)^2}{4} \|r\omega_2^{(p-1)}\|_N^2 \\ &\quad - \|\omega_1^{(p)}\|_N^2 - r^2 \|d_N \omega_2^{(p-1)}\|_N^2 \\ &\quad - \|\delta_N \omega_1^{(p)}\|_N^2 - \|r\omega_2^{(p-1)}\|_N^2 \} \\ &\geq C \int_0^\infty r^{m-2(p+1)} \{ \|\omega_1^{(p)}\|_N^2 + \|r\omega_2^{(p-1)}\|_N^2 \} dr \\ &= C \|r^{-1} f\|_{C(N)}^2. \end{aligned}$$

where $C = \min\{(n-p-1)^2 - 1, (n-p+1)^2 - 1\}$. Since $p \neq n-1, n, n+1$, $C \geq 3$ and we have the desired inequality.

Proposition A.2. *For $t \geq 2$ we define a function*

$$(A.10)\qquad \phi_n(t) := \frac{n(t-1)^2}{(n-1)^2}\{1 + \frac{1}{t-1}\{n(t-2)\}^{\frac{t-2}{t-1}}\}.$$

Then for $m \geq 2$,

$$(A.11)\qquad n(m-1)^2 \sup_{0 \leq t \leq 1} \frac{1+nt}{1+nt^{m-1}} < (n-1)^2 \phi_n(m).$$

Proof. Set $F_m(t) := (1+nt)/(1+nt^m)$ for $t \in [0,1]$. Let $\alpha \in (0,1)$ be the number such that $F_m(\alpha) = \max_{t\in[0,1]} F_m(t)$. Since $F'_m(\alpha) = 0$, α satisfies

$$n(m+1)\alpha^m + m\alpha^{m-1} - 1 = 0. \tag{A.12}$$

Therefore

$$F_m(\alpha) = 1 + \frac{n(m-1)}{m}\alpha. \tag{A.13}$$

Set $\beta := \{n(m-1)\}^{-\frac{1}{m}}$. Then, β is a solution of $f(t) := n(m-1)t^m - 1 = 0$. Since $f(\alpha) < 0$, we have $\alpha < \beta$. Therefore (A.13) yields

$$F_m(\alpha) < 1 + \frac{1}{m}\{n(m-1)\}^{\frac{m-1}{m}} = \frac{(n-1)^2}{nm^2}\phi_n(m+1). \tag{A.14}$$

Hence (A.11) follows from (A.14).

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